

FOUR-MANIFOLDS ADMITTING HYPERELLIPTIC BROKEN LEFSCHETZ FIBRATIONS

KENTA HAYANO AND MASATOSHI SATO

ABSTRACT. We introduce hyperelliptic simplified (more generally, directed) broken Lefschetz fibrations, which is a generalization of hyperelliptic Lefschetz fibrations. We construct involutions on the total spaces of such fibrations of genus $g \geq 3$ and extend these involutions to the four-manifolds obtained by blowing up the total spaces. The extended involutions induce double branched coverings over blown up sphere bundles over the sphere. We also show that the regular fiber of such a fibration of genus $g \geq 3$ represents a non-trivial rational homology class of the total space.

1. INTRODUCTION

A broken Lefschetz fibration is a smooth map from a four-manifold to a surface which has at most two types of singularities, called Lefschetz singularity and indefinite fold singularity. This fibration was introduced in [1] as a fibration structure compatible with near-symplectic structures.

A simplified broken Lefschetz fibration is a broken Lefschetz fibration over the sphere which satisfies several conditions on fibers and singularities. This fibration was first defined by Baykur [3]. In spite of the strict conditions in the definition of this fibration, it is known that every closed oriented four-manifold admits a simplified broken Lefschetz fibration (this fact follows from the result of Williams [15] together with a certain move of singularities defined by Lekili [12]). For a simplified broken Lefschetz fibration, we can define a monodromy representation of this fibration as we define for a Lefschetz fibration. So we can define hyperelliptic simplified broken Lefschetz fibrations as a generalization of hyperelliptic Lefschetz fibrations. Hyperelliptic Lefschetz fibrations have been studied in many fields, algebraic geometry and topology for example, and it has been shown that the total spaces of such fibrations satisfy strong conditions on the signature, the Euler characteristic and so on (see e.g. [9]). So it is natural to ask how far total spaces of hyperelliptic simplified broken Lefschetz fibrations are restricted or what conditions these spaces satisfy. The following result gives a partial answer of these questions:

Theorem 1.1. *Let $f : M \rightarrow S^2$ be a genus- g hyperelliptic simplified broken Lefschetz fibration. We assume that g is greater than or equal to 3.*

- (i) *Let s be the number of Lefschetz singularities of f whose vanishing cycles are separating. Then there exists an involution*

$$\omega : M \rightarrow M$$

such that the fixed point set of ω is the union of (possibly nonorientable) surfaces and s isolated points. Moreover, ω can be extended to an involution

$$\bar{\omega} : M \#_s \overline{\mathbb{CP}^2} \rightarrow M \#_s \overline{\mathbb{CP}^2}$$

such that $M_{\#s\overline{\mathbb{CP}^2}}/\overline{\omega}$ is diffeomorphic to $S_{\#2s\overline{\mathbb{CP}^2}}$, where S is S^2 -bundle over S^2 , and the quotient map

$$/\overline{\omega} : M_{\#s\overline{\mathbb{CP}^2}} \rightarrow M_{\#s\overline{\mathbb{CP}^2}}/\overline{\omega} \cong S_{\#2s\overline{\mathbb{CP}^2}}$$

is the double branched covering.

- (ii) Let $F \in M$ be a regular fiber of f . Then F represents a non-trivial rational homology class of M , that is, $[F] \neq 0$ in $H_2(M; \mathbb{Q})$.

Remark 1.2. Theorem 1.1 can be generalized to directed broken Lefschetz fibrations, which are broken Lefschetz fibrations over the sphere satisfying certain conditions on singularities (cf. Theorem 4.7).

The statement (i) in Theorem 1.1 is a generalization of the results of Fuller [7] and Siebert-Tian [14] on hyperelliptic Lefschetz fibrations. Indeed, they proved independently that, after blowing up s times, the total space of a hyperelliptic Lefschetz fibration (with arbitrary genus) is a double branched covering of a manifold obtained by blowing up a sphere bundle over the sphere $2s$ times, where s is the number of Lefschetz singularities of the fibration whose vanishing cycles are separating. Fuller used handle decompositions and Kirby diagrams to prove the above statement, while Siebert and Tian used complex geometrical techniques. We also use handle decompositions to prove the statement (i) in Theorem 1.1, but our method is slight different from the one Fuller used; ours can give an involution of the total space of a fibration explicitly and this explicit description is used in the proof of the statement (ii) in Theorem 1.1.

Auroux, Donaldson and Katzarkov [1] showed that a closed oriented four-manifold M admits a near-symplectic form if and only if M admits a broken Lefschetz pencil (or fibration) f which has a cohomology class $h \in H^2(M)$ such that $h(\Sigma) > 0$ for every connected component of every fiber of f . Moreover, for a broken Lefschetz fibration f satisfying the cohomological condition above, we can take a near-symplectic form θ so that all the fibers of f are symplectic outside of the singularities. Since every fiber of a simplified broken Lefschetz fibration is connected, we obtain the following corollary.

Corollary 1.3. *Let $f : M \rightarrow S^2$ be a hyperelliptic simplified broken Lefschetz fibration of genus $g \geq 3$. Then there exists a near-symplectic form θ on M which makes all the fibers of f symplectic outside of the singularities.*

Since the self-intersection of a regular fiber of a broken Lefschetz fibration is equal to 0, we also obtain:

Corollary 1.4. *A closed oriented four-manifold with definite intersection form cannot admit any hyperelliptic simplified broken Lefschetz fibrations of genus $g \geq 3$.*

We emphasize that the condition $g \geq 3$ in the above statements is essential. Indeed, it is proved in [1] that S^4 and $\#n\overline{\mathbb{CP}^2}$ ($n \geq 1$) admit genus-1 simplified broken Lefschetz fibrations. Since every simplified broken Lefschetz fibration with genus less than 3 is hyperelliptic, these examples mean that Corollary 1.3 and 1.4 do not hold without the assumption $g \geq 3$.

It is shown in [10] that a simply connected four-manifold with a positive definite intersection form cannot admit any genus-1 simplified broken Lefschetz fibrations except S^4 . In particular, $\#n\overline{\mathbb{CP}^2}$ ($n \geq 1$) cannot admit any genus-1 simplified broken Lefschetz fibrations. However, we prove the following theorem.

Theorem 1.5. *For each $n \geq 0$, $\sharp n\mathbb{CP}^2$ admits a genus-2 simplified broken Lefschetz fibration.*

The above theorem also means that Corollary 1.4 does not hold without the assumption on genus. Moreover, it is easy to see that the fibration in Theorem 1.5 cannot be compatible with any near-symplectic forms although $\sharp n\mathbb{CP}^2$ ($n \geq 1$) admits a near-symplectic form.

In general, a genus- g simplified broken Lefschetz fibration can be changed into a genus- $(g+1)$ simplified broken Lefschetz fibration by a certain homotopy of fibrations, called *flip and slip* (for the detail of this homotopy, see e.g. [2]). Therefore, for any $g \geq 3$, we can easily construct genus- g simplified broken Lefschetz fibrations on S^4 , $\sharp n\mathbb{CP}^2$ and $\sharp n\overline{\mathbb{CP}}^2$ ($n \geq 1$). However, these fibrations are not hyperelliptic because of Corollary 1.4.

In Section 2, we review the definitions of broken Lefschetz fibrations and simplified ones. We also review the basic properties of monodromy representations of broken Lefschetz fibrations. After reviewing the hyperelliptic mapping class group, we give the definition of hyperelliptic simplified broken Lefschetz fibrations. In Section 3, we prove a certain Lemma on the subgroup of the hyperelliptic mapping class group which consists of elements preserving a simple closed curve c . This lemma plays a key role in the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.1. In Section 5, we construct a genus-2 simplified broken Lefschetz fibration on $\sharp n\mathbb{CP}^2$ ($n \geq 1$) to prove Theorem 1.5.

2. PRELIMINARIES

2.1. Broken Lefschetz fibrations. We first give the precise definition of broken Lefschetz fibrations.

Definition 2.1. Let M and Σ be compact oriented smooth manifolds of dimension 4 and 2, respectively. A smooth map $f : M \rightarrow \Sigma$ is called a *broken Lefschetz fibration* (BLF, for short) if it satisfies the following conditions:

- (1) $f^{-1}(\partial\Sigma) = \partial M$;
- (2) f has at most the two types of singularities which is locally written as follows:
 - $(z_1, z_2) \mapsto \xi = z_1 z_2$, where (z_1, z_2) (resp. ξ) is a complex local coordinate of M (resp. Σ) compatible with its orientation;
 - $(t, x_1, x_2, x_3) \mapsto (y_1, y_2) = (t, x_1^2 + x_2^2 - x_3^2)$, where (t, x_1, x_2, x_3) (resp. (y_1, y_2)) is a real coordinate of M (resp. Σ).

The first singularity in the condition (2) of Definition 2.1 is called a *Lefschetz singularity* and the second one is called an *indefinite fold singularity*. We denote by \mathcal{C}_f the set of Lefschetz singularities of f and by Z_f the set of indefinite fold singularities of f . We remark that a Lefschetz fibration is a BLF which has no indefinite fold singularities.

Let $f : M \rightarrow S^2$ be a BLF over the 2-sphere. Suppose that the restriction of f to the set of singularities is injective and that the image $f(Z_f)$ is the disjoint union of embedded circles parallel to the equator of S^2 . We put $f(Z_f) = Z_1 \amalg \cdots \amalg Z_m$, where Z_i is the embedded circle in S^2 . We choose a path $\alpha : [0, 1] \rightarrow S^2$ satisfying the following properties:

- (1) $\text{Im } \alpha$ is contained in the complement of $f(\mathcal{C}_f)$;
- (2) α starts at the south pole $p_s \in S^2$ and connects the south pole to the north pole $p_n \in S^2$;
- (3) α intersects each component of $f(Z_f)$ at one point transversely.

We put $\{q_i\} = Z_i \cap \text{Im } \alpha$ and $\alpha(t_i) = q_i$. We assume that q_1, \dots, q_m appear in this order when we go along α from p_s to p_n (see Figure 1).

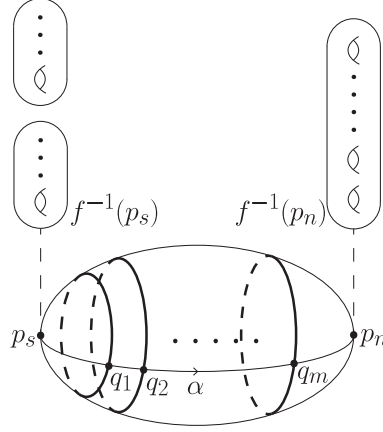


FIGURE 1. The example of the path α . The bold circles describe $f(Z_f)$.

The preimage $f^{-1}(\text{Im } \alpha)$ is a 3-manifold which is a cobordism between $f^{-1}(p_s)$ and $f^{-1}(p_n)$. By the local coordinate description of the indefinite fold singularity, it is easy to see that $f^{-1}(\alpha([0, t_i + \varepsilon]))$ is obtained from $f^{-1}(\alpha([0, t_i - \varepsilon]))$ by either 1 or 2-handle attachment for each $i = 1, \dots, m$. In particular, we obtain a handle decomposition of the cobordism $f^{-1}(\text{Im } \alpha)$.

Definition 2.2. A BLF f is said to be *directed* if it satisfies the following conditions:

- (1) the restriction of f to the set of singularities is injective and the image $f(Z_f)$ is the disjoint union of embedded circles parallel to the equator of S^2 ;
- (2) all the handles in the above handle decomposition of $f^{-1}(\text{Im } \alpha)$ is index-1;
- (3) all Lefschetz singularities of f are in the preimage of the component of $S^2 \setminus (Z_1 \amalg \dots \amalg Z_m)$ which contains the point p_n .

The third condition in the above definition is not essential. Indeed, we can change a BLF f which satisfies the conditions (1) and (2) so that it satisfies the condition (3) (cf. Baykur [3]).

For a directed BLF f , we assume that the set of indefinite fold singularities of f is connected and that all the fibers of f are connected. We call such a BLF a *simplified broken Lefschetz fibration*. For a simplified BLF f , Z_f is empty set or an embedded circle in M . If Z_f is not empty, the image $f(Z_f)$ is an embedded circle in S^2 . So $S^2 \setminus \text{Int } \nu f(Z_f)$ consists of two 2-disks D_1 and D_2 and the genus of the regular fiber of the fibration $\text{res } f : f^{-1}(D_1) \rightarrow D_1$ is just one higher than that of the fibration $\text{res } f : f^{-1}(D_2) \rightarrow D_2$. we call $f^{-1}(D_1)$ (resp. $f^{-1}(D_2)$) the *higher side* (resp. *lower side*) of f and $f^{-1}(\nu f(Z_f))$ the *round cobordism* of f . By the definition, all the Lefschetz singularities of f are in the higher side of f . We call the genus of the regular fiber in the higher side the *genus* of f .

In this paper, we will use the abbreviation SBLF to refer to a simplified BLF.

2.2. Monodromy representations. Let $f : M \rightarrow B$ be a genus- g Lefschetz fibration. We denote by $\mathcal{C}_f = \{z_1, \dots, z_n\}$ the set of Lefschetz singularities of f and put $y_i = f(z_i)$. For a base point $y_0 \in B \setminus f(\mathcal{C}_f)$, a homomorphism $\varrho_f : \pi_1(B \setminus f(\mathcal{C}_f), y_0) \rightarrow \mathcal{M}_g$, called a *monodromy representation* of f , can be defined, where $\mathcal{M}_g = \text{Diff}^+ \Sigma_g / \text{Diff}_0^+ \Sigma_g$ is the mapping class group of the closed oriented surface Σ_g . We endow the C^∞ topology with $\text{Diff}^+ \Sigma_g$ and then $\text{Diff}_0^+ \Sigma_g$ is the component of $\text{Diff}^+ \Sigma_g$ containing the identity map. (the reader should refer to [8] for the precise definition of this homomorphism).

We look at the case $B = D^2$. For each $i = 1, \dots, n$, we take an embedded path $\alpha_i \subset D^2$ satisfying the following conditions:

- each α_i connects y_0 to y_i ;
- $\alpha_i \cap f(\mathcal{C}_f) = \{y_i\}$;
- $\alpha_i \cap \alpha_j = \{y_0\}$, for all $i \neq j$;
- $\alpha_1, \dots, \alpha_n$ appear in this order when we travel counterclockwise around y_0 .

For each $i = 1, \dots, n$, we denote by $a_i \in \pi_1(D^2 \setminus f(\mathcal{C}_f), y_0)$ the element represented by the loop obtained by connecting a counterclockwise circle around y_i to y_0 by using α_i . We put $W_f = (\varrho_f(a_1), \dots, \varrho_f(a_n)) \in \mathcal{M}_g^n$. This sequence is called a *Hurwitz system* of f . By the conditions on paths $\alpha_1, \dots, \alpha_n$, the product $\varrho_f(a_1) \cdots \varrho_f(a_n)$ is equal to the monodromy of the boundary of D^2 . It is known that each $\varrho_f(a_i)$ is the right-handed Dehn twist along a certain simple closed curve c_i , called a *vanishing cycle* of the Lefschetz singularity z_i (cf. [11] or [13]).

Remark 2.3. W_f is not unique for f . Indeed, it depends on the choice of paths $\alpha_1, \dots, \alpha_n$ and the choice of the identification of $f^{-1}(y_0)$ with the closed oriented surface Σ_g . However, it is known that another Hurwitz system \tilde{W}_f is obtained from W_f by successive application of the following transformations:

- $(\dots, g_i, g_{i+1}, \dots) \mapsto (\dots, g_{i+1}, g_{i+1}^{-1}g_i g_{i+1}, \dots)$ and its inverse transformation;
- $(g_1, \dots, g_n) \mapsto (h^{-1}g_1 h, \dots, h^{-1}g_n h)$,

where $g_i, h \in \mathcal{M}_g$ (cf. [8]). Two sequences of elements in \mathcal{M}_g is called *Hurwitz equivalent* if one is obtained from the other by successive application of the transformations above

Let $\hat{f} : M \rightarrow S^2$ be a genus- g SBLF with non-empty indefinite fold singularities. We denote by M_h the higher side of \hat{f} . Then the restriction $\text{res}\hat{f} : M_h \rightarrow D^2$ is a Lefschetz fibration over D^2 . So a monodromy representation and a Hurwitz system of $\text{res}\hat{f}$ can be defined and are called a *monodromy representation* and a *Hurwitz system* of \hat{f} , respectively. We denote them by $\varrho_{\hat{f}}$ and $W_{\hat{f}}$. For the Lefschetz fibration $\text{res}\hat{f} : M_h \rightarrow D^2$, we choose a base point y_0 and paths $\alpha_1, \dots, \alpha_n$ as in the preceding paragraph. We also take a path $\alpha : [0, 1] \rightarrow S^2$ satisfying the following conditions:

- α starts at the base point y_0 and connects y_0 to a point in the image of the lower side of \hat{f} ;
- for each $i = 1, \dots, n$, $\alpha \cap \alpha_i = \{y_0\}$;
- α intersects the image $\hat{f}(Z_{\hat{f}})$ at one point transversely;
- $\text{Im } \alpha \cap \hat{f}(\mathcal{C}_{\hat{f}}) = \emptyset$;
- $\alpha_1, \dots, \alpha_n, \alpha$ appear in this order when we travel counterclockwise around y_0 .

We put $q = \alpha(t) \in \text{Im } \alpha \cap \hat{f}(Z_{\hat{f}})$. The preimage $\hat{f}^{-1}(\alpha([0, t + \varepsilon]))$ is obtained from the preimage $\hat{f}^{-1}(\alpha([0, t - \varepsilon])) \cong \hat{f}^{-1}(p_0) \times [0, t - \varepsilon]$ by the 2-handle attachment. We regard the attaching circle c of the 2-handle as a simple closed curve in $\hat{f}^{-1}(p_0) \cong \Sigma_g$. We call this simple closed curve a *vanishing cycle* of the indefinite fold singularity of \hat{f} .

Lemma 2.4 (Baykur [3]). *The product $\varrho_{\hat{f}}(a_1) \cdots \varrho_{\hat{f}}(a_n)$ is contained in $\mathcal{M}_g(c)$, where $\mathcal{M}_g(c)$ is the subgroup of the group \mathcal{M}_g which consists of elements represented by a map preserving the curve c , that is,*

$$\mathcal{M}_g(c) = \{[T] \in \mathcal{M}_g \mid T(c) = c\}.$$

For an element $\psi \in \mathcal{M}_g(c)$, we take a representative $T \in \psi$ so that T preserves the curve c . Then T induces the homeomorphism $T : \Sigma_g \setminus c \rightarrow \Sigma_g \setminus c$ and this homeomorphism can be extended to the

homeomorphism $\hat{T} : \Sigma_{g-1} \rightarrow \Sigma_{g-1}$ by regarding $\Sigma_g \setminus c$ as the genus- $(g-1)$ surface with two punctures. Eventually, we can define the homomorphism Φ_c as follows:

$$\begin{array}{ccc} \Phi_c : \mathcal{M}_g(c) & \longrightarrow & \mathcal{M}_{g-1} \\ \Psi & & \Psi \\ \psi = [T] & \longmapsto & [\hat{T}]. \end{array}$$

Remark 2.5. Let $c \subset \Sigma_g$ be a separating simple closed curve. We can regard $\Sigma_g \setminus c$ as the disjoint union of the two once punctured surfaces of genus h and $g-h$. So we can define the homomorphism $\Phi_c : \mathcal{M}_g(c) \rightarrow \mathcal{M}_h \times \mathcal{M}_{g-h}$ as we define Φ_c for a non-separating curve c , where $\mathcal{M}_g(c^{\text{ori}})$ is the subgroup of $\mathcal{M}_g(c)$ which consists of elements represented by maps preserving c and its orientation.

Lemma 2.6 (Baykur [3]). *The product $\varrho_{\hat{f}}(a_1) \cdots \varrho_{\hat{f}}(a_n)$ is contained in the kernel of Φ_c . Conversely, if simple closed curves $c, c_1, \dots, c_n \subset \Sigma_g$ satisfy the following conditions:*

- c is non-separating;
- $t_{c_1} \cdots t_{c_n} \in \text{Ker } \Phi_c$,

then there exists a genus- g SBLF $f : M \rightarrow S^2$ such that $W_f = (t_{c_1}, \dots, t_{c_n})$ and a vanishing cycle of the indefinite fold of f is c .

2.3. The hyperelliptic mapping class group. Let Σ_g be a closed oriented surface of genus $g \geq 1$. Denote by $\iota : \Sigma_g \rightarrow \Sigma_g$ an involution described in Figure 2.

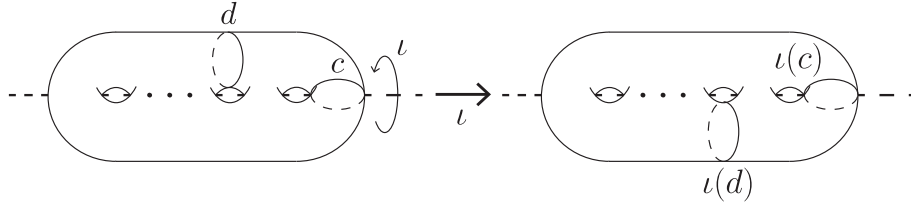


FIGURE 2. the hyperelliptic involution on the surface Σ_g .

Let $C(\iota)$ denote the centralizer of ι in the diffeomorphism group $\text{Diff}_+ \Sigma_g$, and endow $C(\iota) \subset \text{Diff}_+ \Sigma_g$ with the relative topology. The inclusion homomorphism $C(\iota) \rightarrow \text{Diff}_+ \Sigma_g$ induces a natural homomorphism $\pi_0 C(\iota) \rightarrow \mathcal{M}_g$ between their path-connected components.

Theorem 2.7 (Birman-Hilden [5]). *When $g \geq 2$, the homomorphism $\pi_0 C(\iota) \rightarrow \mathcal{M}_g$ is injective.*

Denote the image of this homomorphism by \mathcal{H}_g for $g \geq 1$. This group is called the *hyperelliptic mapping class group*. In fact, they showed the above result in more general settings later. See [5] for details.

A Lefschetz fibration is said to be *hyperelliptic* if we can take an identification of the fiber of a base point with the closed oriented surface so that the image of the monodromy representation of the fibration is contained in the hyperelliptic mapping class group. So it is natural to generalize this definition to directed (and especially simplified) BLFs as follows: Let $f : M \rightarrow S^2$ be a directed BLF. We use the same notations as we use in the argument before Definition 2.2. We take a disk neighborhood $D \subset S^2 \setminus f(Z_f)$ of p_n so that $f(\mathcal{C}_f)$ is contained in D . We put $r_i = \alpha \left(\frac{t_i + t_{i+1}}{2} \right)$ ($i = 1, \dots, m-1$) and $r_m = p_n$. Let $d_i \subset f^{-1}(r_i)$ be the vanishing cycle of Z_i determined by α . Once we fix an identification of $f^{-1}(r_m)$ with $\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_k}$, we obtain an involution ι_i on $f^{-1}(r_i)$

induced by the hyperelliptic involution on $f^{-1}(r_m)$ since we can identify $f^{-1}(r_{i-1}) \setminus \{\text{two points}\}$ with $f^{-1}(r_i) \setminus d_i$ by using α . f is said to be *hyperelliptic* if it satisfies the following conditions for a suitable identification of $f^{-1}(r_m)$ with $\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_k}$:

- the image of the monodromy representation of the Lefschetz fibration $\text{res } f : f^{-1}(D) \rightarrow D$ is contained in the group \mathcal{H}_g ;
- d_i is preserved by the involution ι_i up to isotopy.

In this paper, we will call a hyperelliptic SBLF HSBLF for short.

Remark 2.8. Every SBLF whose genus is less than or equal to 2 is hyperelliptic since $\mathcal{H}_g = \mathcal{M}_g$ and all simple closed curves in Σ_g is preserved by ι if $g \leq 2$.

2.4. Handle decompositions. Let $f : M \rightarrow S^2$ be a genus- g SBLF and M_h (resp. M_r and M_l) the higher side (resp. the round cobordism and the lower side) of f . Then $\text{res } f : M_h \rightarrow D^2$ is a Lefschetz fibration over the disk. We choose $y_0 \in D^2$ and $\alpha_1, \dots, \alpha_n \subset D^2$ as in subsection 2.2. Let $D \subset \text{Int } D^2 \setminus \mathcal{C}_f$ be a disk whose boundary intersects each path α_i at one point transversely. Denote by $w_i \in \partial D$ the intersection between ∂D and α_i and by $c_i \subset f^{-1}(w_i)$ a vanishing cycle of the Lefschetz singularity in the fiber $f^{-1}(y_i)$.

Theorem 2.9 (Kas [11]). *M_h is obtained by attaching n 2-handles to $f^{-1}(D) \cong D \times \Sigma_g$ whose attaching circles are c_1, \dots, c_n and framings of these handles are -1 relative to the framing along the fiber.*

We call $R^\lambda = [0, 1] \times D^\lambda \times D^{3-\lambda} / ((1, x_1, x_2, x_3) \sim (0, \pm x_1, x_2, \pm x_3))$ a (4-dimensional) *round λ -handle* ($\lambda = 1, 2$) and $X^4 \cup_\varphi R^\lambda$ a 4-manifold obtained by attaching a round λ -handle to the 4-manifold X^4 , where $\varphi : [0, 1] \times \partial D^\lambda \times D^{3-\lambda} / \sim \rightarrow \partial X$ is an embedding. A round handle R^λ is said to be *untwisted* if the sign in the equivalence relation is plus and *twisted* otherwise.

Theorem 2.10 (Baykur [3]). *$M_h \cup M_r$ is obtained by attaching a round 2-handle to M_h . Moreover, a circle $\{t\} \times \partial D^2 \times \{0\}$ in the attaching region of R^2 is attached along a vanishing cycle of indefinite fold singularities of f .*

We remark that the isotopy class of the attaching map $\varphi : [0, 1] \times \partial D^2 \times D^1 / \sim \rightarrow \partial M_h$ is uniquely determined by a vanishing cycle of indefinite fold of f if the genus of f is greater than or equal to 2. In particular, the total space of f is uniquely determined by a vanishing cycle of indefinite fold of f and ones of Lefschetz singularities of f . if the genus of f is greater than or equal to 3. However, there exist infinitely many SBLFs of genus $g \leq 2$ such that they have the same vanishing cycles but the total spaces of them are mutually distinct (see [4] or [10]).

Round 2-handle attachment is given by 2-handle attachment followed by 3-handle attachment (cf. [3]). So we obtain a handle decomposition of $M_h \cup M_r$ by the above theorems. Since M_l contains no singularities of f , the map $\text{res } f : M_l \rightarrow D^2$ is the trivial Σ_{g-1} -bundle. In particular, $M_l \cong D^2 \times \Sigma_{g-1}$ and we obtain a handle decomposition of $M = M_h \cup M_r \cup M_l$. Moreover, we can draw a Kirby diagram of M by the decomposition (For more details on this decomposition and corresponding diagram, see [3]).

We also obtain a handle decomposition of the total space of a directed BLF $f : M \rightarrow S^2$ by the same argument as above. Indeed, we can decompose M into $D^2 \times (\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_m})$, n_1 2-handles, n_2 round 2-handles and $D^2 \times (\Sigma_{h_1} \amalg \cdots \amalg \Sigma_{h_m})$, where n_1 is the number of the Lefschetz singularities of f and n_2 is the number of the components of the set of indefinite fold singularities of f .

3. A SUBGROUP $\mathcal{H}_g(c)$ OF THE HYPERELLIPTIC MAPPING CLASS GROUP WHICH PRESERVES A CURVE c

Let c be an essential simple closed curve in the surface Σ_g which is preserved by the involution $\iota \in \text{Diff}_+ \Sigma_g$ as a set. Let $\mathcal{H}_g(c)$ denote the subgroup of the hyperelliptic mapping class group defined by $\mathcal{H}_g(c) := \mathcal{H}_g \cap \mathcal{M}_g(c)$. As introduced in Theorem 2.7, the hyperelliptic mapping class group \mathcal{H}_g is isomorphic to the group consisting of the path-connected components of $C(\iota)$. Hence, the group $\mathcal{H}_g(c)$ consists of the mapping classes which can be represented by both of elements in the centralizer $C(\iota)$ and elements in $\text{Diff}_+(\Sigma_g, c)$. Let $\mathcal{H}_g^s(c)$ denote the subgroup of $\pi_0 C(\iota)$ defined by $\mathcal{H}_g^s(c) := \{[T] \in \pi_0 C(\iota) \mid T(c) = c\}$. In this section, we will prove the following lemma.

Lemma 3.1. *Let $g \geq 2$. The natural isomorphism $\pi_0 C(\iota) \rightarrow \mathcal{H}_g$ in Theorem 2.7 restricts to an isomorphism between the groups $\mathcal{H}_g^s(c)$ and $\mathcal{H}_g(c)$.*

To prove the lemma, It is enough to show that the homomorphism maps $\mathcal{H}_g^s(c)$ onto $\mathcal{H}_g(c)$. Let $[T]$ be a mapping class in $\mathcal{H}_g(c)$. We can choose a representative $T : \Sigma_g \rightarrow \Sigma_g$ in the centralizer $C(\iota)$. Since it is isotopic to some diffeomorphism on Σ_g which preserves the curve c , the curve $T(c)$ is isotopic to c .

We call an isotopy $L_0 : \Sigma_g \times [0, 1] \rightarrow \Sigma_g$ is symmetric if and only if $L_0(*, t) \in C(\iota)$ for any $t \in [0, 1]$. In the following, we will construct a symmetric isotopy $L : \Sigma_g \times [0, 1] \rightarrow \Sigma_g$ satisfying

$$L(*, 0) = T, \text{ and } L(c, 1) = c \subset \Sigma_g.$$

It indicates that $L(*, 1)$ represents an element in $\mathcal{H}_g^s(c)$, and $[L(*, 1)] = [T] \in \pi_0 C(\iota)$. Hence, we see that the homomorphism $\mathcal{H}_g^s(c) \rightarrow \mathcal{H}_g(c)$ is surjective.

To construct the symmetric isotopy $L : \Sigma_g \times [0, 1] \rightarrow \Sigma_g$, we need a proposition, so called the bigon criterion.

Proposition 3.2 (Farb-Margalit Proposition 1.7 [6]). *Let S be a compact surface. The geometric intersection number of two transverse simple closed curves in S is minimal if and only if they do not form a bigon.*

We may assume that the curves c and $T(c)$ are transverse by changing the diffeomorphism T in terms of some symmetric isotopy. Since c and $T(c)$ are isotopic, the minimal intersection number of them is 0. Hence, there exist bigons such that each of their boundaries is the union of an arc of c and an arc of $T(c)$. Choose an innermost bigon Δ among them.

Let α be the arc $c \cap \partial\Delta$ and β the arc $T(c) \cap \partial\Delta$, respectively. Since Δ is a bigon, the endpoints of them coincide. Denote them by $\{x_1, x_2\} \subset \partial\Delta$.

Lemma 3.3.

$$\text{Int } \Delta \cap (T(c) \cup c) = \emptyset$$

Proof. If the set $\text{Int } \Delta \cap c$ is non-empty, there exists an arc of c in Δ which forms a bigon with the arc β . Since the bigon Δ is innermost, it is a contradiction. We can also show that $\text{Int } \Delta \cap T(c) = \emptyset$ in the same way. \square

Note that the bigon $\iota(\Delta)$ is also innermost. By Lemma 3.3, we have $\Delta \cap \iota(\Delta) = \partial\Delta \cap \partial\iota(\Delta)$.

Lemma 3.4.

$$\partial\Delta \cap \partial\iota(\Delta) \subset \{x_1, x_2\}.$$

Proof. Since $\partial\alpha = \partial\beta = \alpha \cap \beta = \{x_1, x_2\}$, it suffices to show that $\text{Int } \alpha \cap \partial\iota(\Delta) = \text{Int } \beta \cap \partial\iota(\Delta) = \emptyset$. Since $\alpha \cap T(c) = \{x_1, x_2\}$, we have $\text{Int } \alpha \cap \iota(\beta) = \emptyset$. Next, we will show that $\text{Int } \alpha \cap \text{Int } \iota(\alpha) = \emptyset$. We assume $\text{Int } \alpha \cap \text{Int } \iota(\alpha) \neq \emptyset$. Since c is simple and contains α and $\iota(\alpha)$, α and $\iota(\alpha)$ must coincide. In particular, we have $\partial\alpha = \partial\iota(\alpha)$. So $\beta \cup \iota(\beta)$ forms a simple closed curve, and this curve is null-homotopic because both of the arcs β and $\iota(\beta)$ are homotopic to $\alpha = \iota(\alpha)$ relative to their boundaries. Since $T(c)$ is simple and contains β and $\iota(\beta)$, $T(c)$ and $\beta \cup \iota(\beta)$ must coincide. This contradicts that $T(c)$ is essential. In the same way, we can show that $\text{Int } \beta \cap \partial\iota(\Delta) = \emptyset$. \square

Let Σ_g^ι denote the fixed point set of the involution ι on Σ_g .

Lemma 3.5. *If c is non-separating, the set $c \cap \Sigma_g^\iota$ consists of 2 points, and*

$$c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota.$$

If c is separating,

$$c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset.$$

Proof. Endow the curves c and $T(c)$ with arbitrary orientations.

First, consider the case when c is a non-separating simple closed curve. In this case, the curve $T(c)$ is also non-separating. They represent nontrivial homology classes in $H_1(\Sigma_g; \mathbf{Z})$. Since the involution ι acts on $H_1(\Sigma_g; \mathbf{Z})$ by -1 , it changes the orientations of c and $T(c)$. Hence, both of the sets $c \cap \Sigma_g^\iota$ and $T(c) \cap \Sigma_g^\iota$ consist of 2 points.

We will show that $T(c) \cap \Sigma_g^\iota = c \cap \Sigma_g^\iota$. Since c and $T(c)$ are isotopic, the Dehn twists t_c and $t_{T(c)}$ represent the same element in \mathcal{H}_g . The mapping classes $\Psi([t_c])$ and $\Psi([t_{T(c)}])$ in \mathcal{M}_0^{2g+2} permute the branched points $p(c \cap \Sigma_g^\iota)$ and $p(T(c) \cap \Sigma_g^\iota)$, respectively. Hence, the sets $p(c \cap \Sigma_g^\iota)$ and $p(T(c) \cap \Sigma_g^\iota)$ coincide. It shows that $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota$.

Next, let c be a separating simple closed curve. Since ι preserves the orientations of the subsurfaces bounded by c or $T(c)$, it also preserves the orientation of c and $T(c)$. In general, if an involution acts on a circle preserving its orientation, it does not have a fixed point. Hence, we have $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset$. \square

Proof of Lemma 3.1. Let c be a non-separating curve. By Lemma 3.5, the geometric intersection number of c and $T(c)$ is at least 2. Hence, there is an innermost bigon Δ . By Lemma 3.4, the cardinality $\sharp(\Delta \cap \iota(\Delta))$ is equal to 0, 1, or 2 as shown in Figure 3.

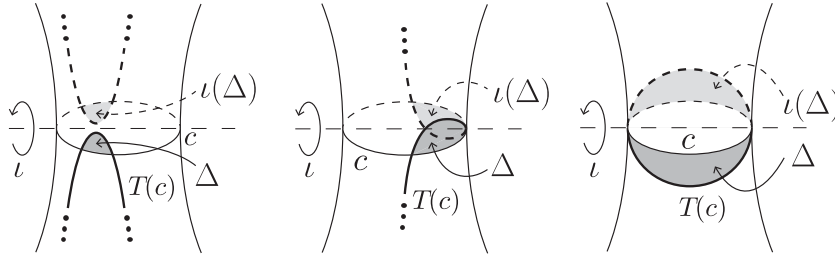


FIGURE 3. Left: $\sharp(\Delta \cap \iota(\Delta)) = 0$. Center: $\sharp(\Delta \cap \iota(\Delta)) = 1$. Right: $\sharp(\Delta \cap \iota(\Delta)) = 2$.

The bold curves describe the curves $T(c)$.

Firstly, assume that $\sharp(\Delta \cap \iota(\Delta)) = 0$. In this case, there is a symmetric isotopy $L_1 : \Sigma_g \times [0, 1] \rightarrow \Sigma_g$ such that $L_1(*, 0)$ is the identity, and $L_1(*, 1)$ collapses the bigon Δ as in Figure 4. Therefore, we can

decrease the geometric intersection number of c and $T(c)$ by 4 by replacing the diffeomorphism T by $L_1(*, 1)T$.

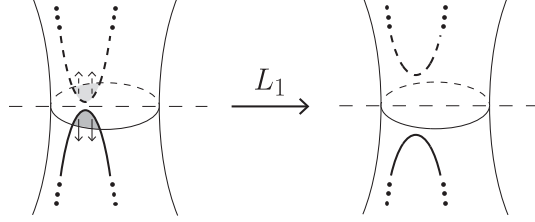


FIGURE 4.

Secondly, assume that $\sharp(\Delta \cap \iota(\Delta)) = 1$. In this case, we also have a symmetric isotopy $L_2 : \Sigma_g \times [0, 1] \rightarrow \Sigma_g$ which decreases the geometric intersection number by 2 as in Figure 5. Note that $\Delta \cap \iota(\Delta)$ is a branched point, and $L_2(*, t)$ fixes it for any $t \in [0, 1]$.

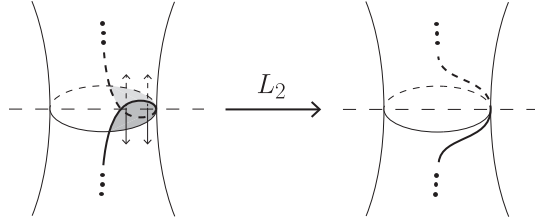


FIGURE 5.

After replacing the diffeomorphism T in these two cases, the branch points $\{x_1, x_2\}$ remains in $c \cap T(c)$. Hence, if we repeat to replace T , the case when $\sharp(\Delta \cap \iota(\Delta)) = 2$ will definitely occur. In this case, there is a symmetric isotopy $L_3 : \Sigma_g \times [0, 1] \rightarrow \Sigma_g$ such that

$$L_3(*, 0) \text{ is the identity map,}$$

$$L_3(\beta, 1) = \alpha,$$

$$L_3(\iota(\beta), 1) = \iota(\alpha),$$

as in Figure 6. It indicates that $L_3(*, 1)T$ preserves the curve c . By combining these isotopies, we have obtained a desired symmetric isotopy.

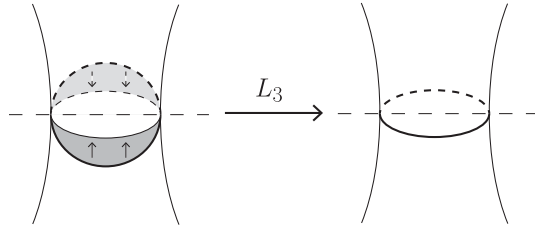


FIGURE 6.

Next, let c be a separating curve. If the geometric intersection number of c and $T(c)$ is 0, the curves c and $T(c)$ bound an annulus A . Since ι acts on A without fixed points, $A/\langle \iota \rangle$ is also an annulus. Hence, we can make a symmetric isotopy which moves $T(c)$ to c .

Assume that the geometric intersection number is not 0. Since we have $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset$, the cardinality $\sharp(\Delta \cap \iota(\Delta)) \neq 1$. By Lemma 3.4, we have $\sharp(\Delta \cap \iota(\Delta)) = 0$ or 2. By the same argument as in the case when c is non-separating, we can collapse the bigons Δ and $\iota(\Delta)$. \square

4. AN INVOLUTION ON HSBLF

In this section, we prove Theorem 1.1.

Proof of (i) in Theorem 1.1. Let $f : M \rightarrow S^2$ be genus- $g \geq 3$ HSBLF, $c_1, \dots, c_n \subset \Sigma_g$ vanishing cycles of Lefschetz singularities of f and $c \subset \Sigma_g$ a vanishing cycle of indefinite fold singularities of f . We assume that c_1, \dots, c_n and c are preserved by the involution $\iota : \Sigma_g \rightarrow \Sigma_g$. By the argument in subsection 2.4, We can decompose M as follows:

$$M = D^2 \times \Sigma_g \cup (h_1^2 \amalg \dots \amalg h_n^2) \cup R^2 \cup D^2 \times \Sigma_{g-1},$$

where $h_i^2 = D^2 \times D^2$ is the 2-handle attached along the simple closed curve $\{p_i\} \times c_i \in \partial D^2 \times \Sigma_g$ and R^2 is a round 2-handle. We first prove existence of an involution ω by using the above decomposition.

Step 1: We define an involution ω_1 on $D^2 \times \Sigma_g$ as follows:

$$\begin{array}{ccc} \omega_1 = \text{id} \times \iota : & D^2 \times \Sigma_g & \longrightarrow & D^2 \times \Sigma_g \\ & \amalg & & \amalg \\ & (z, x) & \longmapsto & (z, \iota(x)). \end{array}$$

In the following steps, we will define an involution on each component in the above decomposition of M which is compatible with the involution ω_1 .

Step 2: We next define an involution $\omega_{2,i}$ on the 2-handle h_i^2 attached along $\{q_i\} \times c_i \subset \partial D^2 \times \Sigma_g$. We will abuse the notation by denoting the attaching circle $\{q_i\} \times c_i$ by c_i .

We take a tubular neighborhood νc_i in $\{q_i\} \times \Sigma_g$ and an identification

$$\nu c_i \cong S^1 \times [-1, 1]$$

so that $c_i = S^1 \times \{0\}$. We assume that the standard orientation of $S^1 \times [-1, 1]$ corresponds to the orientation of $\{q_i\} \times \Sigma_g$. We take a sufficiently small neighborhood I_{q_i} of q_i in ∂D^2 as follows:

$$I_{q_i} = \{q_i \cdot \exp(\sqrt{-1}\theta) \in \partial D^2 \mid \theta \in [-\varepsilon_1, \varepsilon_1]\},$$

where $\varepsilon_1 > 0$ is a sufficiently small number. Moreover, we identify the neighborhood I_{q_i} with the unit interval $[-1, 1]$ by using the following map:

$$\begin{array}{ccc} [-1, 1] & \xrightarrow{\sim} & I_{q_i} \\ \amalg & & \amalg \\ s & \longmapsto & q_i \cdot \exp(\sqrt{-1}\varepsilon_1 s). \end{array}$$

We regard $I_{q_i} \times [-1, 1]$ the subset of \mathbb{C} by the following embedding:

$$\begin{array}{ccc} I_{q_i} \times [-1, 1] & \hookrightarrow & \{z \in \mathbb{C} \mid |\operatorname{Re} z| \leq 1, |\operatorname{Im} z| \leq 1\} \\ \amalg & & \amalg \\ (s, t) & \longmapsto & s + t\sqrt{-1}. \end{array}$$

We put $J = \{z \in \mathbb{C} \mid |\operatorname{Re} z| \leq 1, |\operatorname{Im} z| \leq 1\}$. The orientation of $\partial D^2 \times \Sigma_g$ is reverse to the natural orientation of $J \times S^1$. So the attaching map of the 2-handle h_i^2 is described as follows:

$$\begin{array}{ccc} \varphi_i : \partial D^2 \times D^2 & \longrightarrow & J \times S^1 \subset \partial D^2 \times \Sigma_g \\ \Downarrow & & \Downarrow \\ (w_1, w_2) & \longmapsto & (\varepsilon_2 w_2 w_1, w_1), \end{array}$$

where $\varepsilon_2 > 0$ is a sufficiently small number. We remark that the map φ_i is orientation-preserving if we give the natural orientation of $\partial D^2 \times D^2$.

Case 2.1: If c_i is non-separating, we can take a tubular neighborhood $\nu c_i \cong S^1 \times [-1, 1]$ so that the involution ω_1 acts on νc_i as follows:

$$\begin{array}{ccc} \omega_1|_{\nu c_i} : S^1 \times [-1, 1] & \longrightarrow & S^1 \times [-1, 1] \\ \Downarrow & & \Downarrow \\ (z, t) & \longmapsto & (\bar{z}, -t). \end{array}$$

Since the involution $\omega_1 : D^2 \times \Sigma_g \rightarrow D^2 \times \Sigma_g$ preserves the first component, ω_1 acts on $I_{q_i} \times \nu c_i \cong J \times S^1$ as follows:

$$\begin{array}{ccc} \omega_1|_{J \times S^1} : J \times S^1 & \longrightarrow & J \times S^1 \\ \Downarrow & & \Downarrow \\ (z_1, z_2) & \longmapsto & (\bar{z}_1, \bar{z}_2). \end{array}$$

We define an involution $\omega_{2,i}$ on the 2-handle h_i^2 as follows:

$$\begin{array}{ccc} \omega_{2,i} : D^2 \times D^2 & \longrightarrow & D^2 \times D^2 \\ \Downarrow & & \Downarrow \\ (w_1, w_2) & \longmapsto & (\overline{w_1}, \overline{w_2}). \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} \partial D^2 \times D^2 & \xrightarrow{\omega_{2,i}} & \partial D^2 \times D^2 \\ \varphi_i \downarrow & & \downarrow \varphi_i \\ J \times S^1 & \xrightarrow{\omega_1} & J \times S^1. \end{array}$$

So we can define an involution $\omega_1 \cup \omega_{2,i}$ on the manifold $D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2$.

Case 2.2: If c_i is separating, we can take a tubular neighborhood $\nu c_i \cong S^1 \times [-1, 1]$ so that the involution ω_1 acts on νc_i as follows:

$$\begin{array}{ccc} \omega_1|_{\nu c_i} : S^1 \times [-1, 1] & \longrightarrow & S^1 \times [-1, 1] \\ \Downarrow & & \Downarrow \\ (z, t) & \longmapsto & (-z, t). \end{array}$$

Then ω_1 acts on $I_{q_i} \times \nu c_i \cong J \times S^1$ as follows:

$$\begin{array}{ccc} \omega_1|_{J \times S^1} : J \times S^1 & \longrightarrow & J \times S^1 \\ \Downarrow & & \Downarrow \\ (z_1, z_2) & \longmapsto & (z_1, -z_2). \end{array}$$

We define an involution $\omega_{2,i}$ on the 2-handle h_i^2 as follows:

$$\begin{array}{ccc} \omega_{2,i} : D^2 \times D^2 & \longrightarrow & D^2 \times D^2 \\ \Downarrow & & \Downarrow \\ (w_1, w_2) & \longmapsto & (-w_1, -w_2). \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} \partial D^2 \times D^2 & \xrightarrow{\omega_{2,i}} & \partial D^2 \times D^2 \\ \varphi_i \downarrow & & \downarrow \varphi_i \\ J \times S^1 & \xrightarrow{\omega_1} & J \times S^1. \end{array}$$

So we can define an involution $\omega_1 \cup \omega_{2,i}$ on the manifold $D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2$.

Combining Case 2.1 and Case 2.2, we can define the involution $\tilde{\omega}_2 = \omega_1 \cup (\omega_{2,1} \cup \dots \cup \omega_{2,n})$ on the 4-manifold $M_h = M \cup (h_1^2 \amalg \dots \amalg h_n^2)$. Before giving an involution on the round 2-handle, we look at the Σ_g -bundle structure of ∂M_h . The projection $\pi_h : \partial M_h \rightarrow \partial D^2$ of this bundle is described as follows:

$$\begin{aligned} \pi_h(z, x) &= z & ((z, x) \in \partial D^2 \times \Sigma_g \setminus (\amalg \text{Int } \varphi_i(\partial D^2 \times D^2))), \\ \pi_h(w_1, w_2) &= q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\text{Re } w_1 \text{Re } w_2 - \text{Im } w_1 \text{Im } w_2)) & ((w_1, w_2) \in D^2 \times \partial D^2 \subset \partial h_i^2). \end{aligned}$$

Indeed, the map π_h is well-defined. To see this, we only need to verify the following equation:

$$q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\text{Re } w_1 \text{Re } w_2 - \text{Im } w_1 \text{Im } w_2)) = p_1 \circ \varphi_i(w_1, w_2),$$

where $(w_1, w_2) \in D^2 \times \partial D^2 \subset \partial h_i^2$ and $p_1 : J \times S^1 \rightarrow I_{q_i}$ is the projection. $p_1 \circ \varphi_i(w_1, w_2)$ is calculated as follows:

$$\begin{aligned} p_1 \circ \varphi_i(w_1, w_2) &= p_1(\varepsilon_2 w_2 w_1, w_1) \\ &= q_i \cdot \exp(\sqrt{-1}\varepsilon_1 \text{Re}(\varepsilon_2 w_2 w_1)) \\ &= q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\text{Re } w_1 \text{Re } w_2 - \text{Im } w_1 \text{Im } w_2)) \end{aligned}$$

So we can verify that π_h is well-defined.

Lemma 4.1. *The involution $\tilde{\omega}_2$ preserves the fibers of π_h . Moreover, there exists a lift X of the vector field $\frac{d}{d\theta} \exp(\sqrt{-1}\theta)$ by the map π_h which is compatible with the involution $\tilde{\omega}_2$, that is,*

$$\tilde{\omega}_{2*}(X) = X.$$

Proof of Lemma 4.1. To show that $\tilde{\omega}_2$ preserves the fibers of π_h , it is sufficient to prove $\pi_h \circ \tilde{\omega}_2 = \pi_h$. This equation can be proved easily by direct calculation.

To prove existence of a lift X , we construct X explicitly. We define a vector field X_1 on $\partial D^2 \times \Sigma_g \setminus (\amalg \varphi_i(\partial D^2 \times D^2))$ as follows:

$$X_1(\exp(\sqrt{-1}\theta_0), x) = \left. \frac{d}{d\theta} \exp(\sqrt{-1}\theta) \right|_{\theta=\theta_0} \in T_{(\exp(\sqrt{-1}\theta_0), x)}(\partial D^2 \times \Sigma_g),$$

for a point $(\exp(\sqrt{-1}\theta_0), x) \in \partial D^2 \times \Sigma_g \setminus (\amalg \text{Int } \varphi_i(\partial D^2 \times D^2))$. Then X_1 is described in $J \times S^1$ as follows:

$$X_1(s + t\sqrt{-1}, z) = \left. \frac{1}{\varepsilon_1} \frac{\partial}{\partial s} \right|_s \in T_{(s+t\sqrt{-1}, z)}(J \times S^1).$$

We also define a vector field X_2 on $D^2 \times \partial D^2 \subset \partial h_i^2$ as follows:

$$X_2(w_1, w_2) = \frac{\varrho(|w_1|^2)}{\varepsilon_1 \varepsilon_2 |w_1|^2} \left(x_1 \frac{\partial}{\partial x_2} - y_1 \frac{\partial}{\partial y_2} \right) + \frac{1 - \varrho(|w_1|^2)}{\varepsilon_1 \varepsilon_2} \left(x_2 \frac{\partial}{\partial x_1} - y_2 \frac{\partial}{\partial y_1} \right),$$

where $w_i = x_i + y_i\sqrt{-1}$ and $\varrho : [0, 1] \rightarrow [0, 1]$ is a monotone increasing smooth function which satisfies the following conditions:

- $\varrho(t) = 0$ for $t \in [0, \frac{1}{3}]$;
- $\varrho(t) = 1$ for $t \in [\frac{2}{3}, 1]$.

Then, for $(w_1, w_2) \in \partial D^2 \times \partial D^2$, $d\varphi_i(X_2(w_1, w_2))$ is calculated as follows:

$$\begin{aligned}
& d\varphi_i(X_2(w_1, w_2)) \\
&= d\varphi_i \left(\frac{1}{\varepsilon_1 \varepsilon_2} \left(x_1 \frac{\partial}{\partial x_2} - y_1 \frac{\partial}{\partial y_2} \right) \right) \quad (\because |w_1| = 1) \\
&= \frac{1}{\varepsilon_1 \varepsilon_2} x_1 d\varphi_i \left(\frac{\partial}{\partial x_2} \right) - \frac{1}{\varepsilon_1 \varepsilon_2} y_1 d\varphi_i \left(\frac{\partial}{\partial y_2} \right) \\
&= \frac{1}{\varepsilon_1} x_1 \left(x_1 \frac{\partial}{\partial s} + y_1 \frac{\partial}{\partial t} \right) - \frac{1}{\varepsilon_1} y_1 \left(-y_1 \frac{\partial}{\partial s} + x_1 \frac{\partial}{\partial t} \right) \\
&= \frac{1}{\varepsilon_1} (x_1^2 + y_1^2) \frac{\partial}{\partial s} \\
&= X_1(\varphi_i(w_1, w_2)).
\end{aligned}$$

Hence we can define a vector field $X = X_1 \cup X_2$ on the manifold ∂M_h . Moreover, it can be shown that X_1 and X_2 is a lift of the vector field $\frac{d}{d\theta} \exp(\sqrt{-1}\theta)$ by the map π_h . Thus, the vector field X is a lift of $\frac{d}{d\theta} \exp(\sqrt{-1}\theta)$. We can show that the vector field X is compatible with the involution $\tilde{\omega}_2$ by direct calculation. This completes the proof of Lemma 4.1. \square

We choose a base point $q_0 \in \partial D^2 \setminus (\text{III}_{q_i})$ and define a map $\Theta_X : f^{-1}(q_0) \rightarrow f^{-1}(q_0)$ as follows:

$$\Theta_X(x) = c_{X,x}(2\pi),$$

where $c_{X,x}$ is the integral curve of the vector field X constructed in Lemma 4.1 which satisfies $c_{X,x}(0) = x$. We identify $f^{-1}(q_0)$ with the surface Σ_g via the projection onto the second component. Then the map Θ_X is contained in the centralizer $C(\iota) \subset \text{Diff}_+ \Sigma_g$ since the vector field X is compatible with $\tilde{\omega}_2$. The isotopy class represented by Θ_X is the monodromy of the boundary of M_h . In particular, this class is contained in the group $\mathcal{H}_g(c)$. By Lemma 3.1, there exists an isotopy $H_t : \Sigma_g \rightarrow \Sigma_g$ satisfying the following conditions:

- $H_0 = \Theta_X$;
- H_1 preserves the curve c as a set;
- for each level t , H_t is in the centralizer $C(\iota)$.

Thus, we obtain the following isomorphism as Σ_g -bundles:

$$\partial M_h \cong [0, 1] \times \Sigma_g / ((1, x) \sim (0, H_1(x))).$$

We identify the above Σ_g -bundles via the isomorphism. Then the involution $\tilde{\omega}_2$ acts on ∂M_h as follows:

$$\tilde{\omega}_2(t, x) = (t, \iota(x)),$$

where (t, x) is an element in $[0, 1] \times \Sigma_g / ((1, x) \sim (0, H_1(x))) \cong \partial M_h$.

Step 3: In this step, we define an involution ω_3 on the round 2-handle R^2 . Since c is non-separating and c is preserved by ι , c contains two fixed points of the involution ι . We denote these points by v_1

and v_2 . In addition, we can take a tubular neighborhood $\nu c \cong S^1 \times [-1, 1]$ in Σ_g so that the involution ι acts on νc as follows:

$$\iota(z, t) = (\bar{z}, -t).$$

By perturbing the map H_1 , we can assume that H_1 preserves the neighborhood νc . Then the attaching region of the round 2-handle R^2 is $[0, 1] \times \nu c / ((1, x) \sim (0, H_1(x)))$.

Case 3.1: If H_1 preserves the orientation of c and two points v_1 and v_2 , then the round handle R^2 is untwisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z, t) = (z, t),$$

where $(z, t) \in S^1 \times [-1, 1] \cong \nu c$. Moreover, the attaching map of the round handle is described as follows:

$$\begin{array}{ccc} \varphi : [0, 1] \times \partial D^2 \times D^1 / \sim & \longrightarrow & [0, 1] \times S^1 \times [-1, 1] / \sim \\ \downarrow \Psi & & \downarrow \Psi \\ (s, z, t) & \longmapsto & (s, z, t), \end{array}$$

where $[0, 1] \times \partial D^2 \times D^1$ is the attaching region of R^2 and $[0, 1] \times S^1 \times [-1, 1] \cong [0, 1] \times \nu c$ is the subset of ∂M_h . We define an involution ω_3 on the round handle as follows:

$$\begin{array}{ccc} \omega_3 : [0, 1] \times D^2 \times D^1 / \sim & \longrightarrow & [0, 1] \times D^2 \times D^1 / \sim \\ \downarrow \Psi & & \downarrow \Psi \\ (s, z, t) & \longmapsto & (s, \bar{z}, -t), \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} [0, 1] \times \partial D^2 \times D^1 & \xrightarrow{\omega_3} & [0, 1] \times \partial D^2 \times D^1 \\ \varphi \downarrow & & \downarrow \varphi \\ [0, 1] \times S^1 \times [-1, 1] & \xrightarrow{\tilde{\omega}_2} & [0, 1] \times S^1 \times [-1, 1]. \end{array}$$

Therefore, we obtain an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $M_h \cup M_r = M_h \cup R^2$.

Case 3.2: If H_1 preserves the orientation of c but does not preserve two points v_1 and v_2 , then the round handle R^2 is untwisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z, t) = (-z, t),$$

The attaching map of the round handle is described as follows:

$$\begin{array}{ccc} \varphi : [0, 1] \times \partial D^2 \times D^1 / \sim & \longrightarrow & [0, 1] \times S^1 \times [-1, 1] / \sim \\ \downarrow \Psi & & \downarrow \Psi \\ (s, z, t) & \longmapsto & (s, \exp(\pi\sqrt{-1}s)z, t). \end{array}$$

We define an involution ω_3 on the round handle as follows:

$$\begin{array}{ccc} \omega_3 : [0, 1] \times D^2 \times D^1 / \sim & \longrightarrow & [0, 1] \times D^2 \times D^1 / \sim \\ \downarrow \Psi & & \downarrow \Psi \\ (s, z, t) & \longmapsto & (s, \exp(-2\pi\sqrt{-1}s)\bar{z}, -t), \end{array}$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $M_h \cup M_r = M_h \cup R^2$ by the same reason as in Case 3.1.

Case 3.3: If H_1 does not preserve the orientation of c but preserves two points v_1 and v_2 , then the round handle R^2 is twisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z, t) = (\bar{z}, -t),$$

where $(z, t) \in S^1 \times [-1, 1] \cong \nu c$. Moreover, the attaching map of the round handle is described as follows:

$$\begin{array}{ccc} \varphi : [0, 1] \times \partial D^2 \times D^1 / \sim & \longrightarrow & [0, 1] \times S^1 \times [-1, 1] / \sim \\ \downarrow \Psi & & \downarrow \Psi \\ (s, z, t) & \longmapsto & (s, z, t). \end{array}$$

We define an involution ω_3 on the round handle as follows:

$$\begin{array}{ccc} \omega_3 : [0, 1] \times D^2 \times D^1 / \sim & \longrightarrow & [0, 1] \times D^2 \times D^1 / \sim \\ \downarrow \Psi & & \downarrow \Psi \\ (s, z, t) & \longmapsto & (s, \bar{z}, -t), \end{array}$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $M_h \cup M_r = M_h \cup R^2$.

Case 3.4: If H_1 preserves neither the orientation of c nor two points v_1 and v_2 , then the round handle R^2 is twisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z, t) = (-\bar{z}, -t),$$

where $(z, t) \in S^1 \times [-1, 1] \cong \nu c$. Moreover, the attaching map of the round handle is described as follows:

$$\begin{array}{ccc} \varphi : [0, 1] \times \partial D^2 \times D^1 / \sim & \longrightarrow & [0, 1] \times S^1 \times [-1, 1] / \sim \\ \downarrow \Psi & & \downarrow \Psi \\ (s, z, t) & \longmapsto & (s, \exp(\pi\sqrt{-1}s)z, t). \end{array}$$

We define an involution ω_3 on the round handle as follows:

$$\begin{array}{ccc} \omega_3 : [0, 1] \times D^2 \times D^1 / \sim & \longrightarrow & [0, 1] \times D^2 \times D^1 / \sim \\ \downarrow \Psi & & \downarrow \Psi \\ (s, z, t) & \longmapsto & (s, \exp(-2\pi\sqrt{-1}s)\bar{z}, -t), \end{array}$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $M_h \cup M_r = M_h \cup R^2$.

Eventually, we obtain the involution $\tilde{\omega}_3$ on $M_h \cup M_r$ in any cases. Next we look at Σ_{g-1} -bundle structure of $\partial(M_h \cup M_r)$. The projection $\pi_r : \partial(M_h \cup M_r) \rightarrow [0, 1]/\{0, 1\}$ of this bundle is described as follows:

$$\begin{aligned} \pi_r(s, x) &= s \quad ((s, x) \in ([0, 1] \times \Sigma_g / (1, x) \sim (0, H_1(x))) \setminus ([0, 1] \times \nu c / \sim)); \\ \pi_r(s, z, t) &= s \quad ((s, z, t) \in [0, 1] \times D^2 \times \partial D^1). \end{aligned}$$

Indeed, it is easy to show that π_r is well defined.

Lemma 4.2. *The involution $\tilde{\omega}_3$ preserves the fibers of π_r . Moreover, there exists a lift \tilde{X} of the vector field $\frac{d}{ds}$ on $[0, 1]/\{0, 1\}$ by the map π_r which is compatible with the involution $\tilde{\omega}_3$.*

Proof of Lemma 4.2. It is obvious that the involution $\tilde{\omega}_3$ preserves the fibers of π_r . We construct \tilde{X} as we do in Lemma 4.1. We define a vector field \tilde{X}_1 on $([0, 1] \times \Sigma_g / \sim) \setminus ([0, 1] \times \nu c / \sim)$ as follows:

$$\tilde{X}_1(s, x) = \frac{d}{ds}.$$

We first consider the case H_1 preserves the points v_1 and v_2 . Then we define a vector field \tilde{X}_2 on the round handle R^2 as follows:

$$\tilde{X}_2(s, z, t) = \frac{d}{ds},$$

where $(s, z, t) \in [0, 1] \times D^2 \times \partial D^1 \subset \partial R^2$. Then it is easy to show that $d\varphi\left(\frac{d}{ds}\right) = \frac{d}{ds}$. Hence we can define vector field $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$ on $\partial M_h \cup M_r$. It is obvious that \tilde{X} is a lift of the vector field $\frac{d}{ds}$ on $[0, 1]/\{0, 1\}$ by π_r and is compatible with the involution $\tilde{\omega}_3$.

We next consider the case H_1 does not preserve the points v_1 and v_2 . Then we define a vector field \tilde{X}_2 on R^2 as follows:

$$\tilde{X}_2(s, x + y\sqrt{-1}, t) = \frac{d}{ds} + \pi y \frac{\partial}{\partial x} - \pi x \frac{\partial}{\partial y},$$

where $(s, x + y\sqrt{-1}, t) \in [0, 1] \times D^2 \times \partial D^1 \subset \partial R^2$. The differential $d\varphi(\tilde{X}_2(s, x + \sqrt{-1}y, t))$ is calculated as follows:

$$\begin{aligned} & d\varphi(\tilde{X}_2(s, x + \sqrt{-1}y, t)) \\ &= d\varphi\left(\frac{d}{ds} + \pi y \frac{\partial}{\partial x} - \pi x \frac{\partial}{\partial y}\right) \\ &= \left(\frac{d}{ds} + \pi(-x \sin \pi s - y \cos \pi s) \frac{d}{dx} + \pi(x \cos \pi s - y \sin \pi s) \frac{d}{dy}\right) \\ &\quad + \pi y \left(\cos \pi s \frac{d}{dx} + \sin \pi s \frac{d}{dy}\right) - \pi x \left(-\sin \pi s \frac{d}{dx} + \cos \pi s \frac{d}{dy}\right) \\ &= \frac{d}{ds} \\ &= \tilde{X}_1(\varphi(s, x + \sqrt{-1}y, t)). \end{aligned}$$

Hence we can define a vector field $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$ on $\partial(M_h \cup M_r)$. It is obvious that \tilde{X} is a lift of the vector field $\frac{d}{ds}$ on $[0, 1]/\{0, 1\}$ by π_r . To verify that \tilde{X} is compatible with the involution $\tilde{\omega}_3$, We prove that the following equation holds for any points $x \in \partial(M_h \cup M_r)$:

$$d\tilde{\omega}_3(\tilde{X}(x)) = \tilde{X}(\tilde{\omega}_3(x)).$$

If x is contained in $[0, 1] \times \Sigma_g / \sim \setminus ([0, 1] \times \nu c / \sim)$, the above equation can be proved easily. If $x = (s, x + \sqrt{-1}y, t) \in [0, 1] \times D^2 \times \partial D^1 \subset \partial R^2$, then $d\tilde{\omega}_3(\tilde{X}(x))$ is calculated as follows:

$$\begin{aligned} & d\tilde{\omega}_3(\tilde{X}(x)) \\ &= d\tilde{\omega}_3\left(\frac{d}{ds} + \pi y \frac{\partial}{\partial x} - \pi x \frac{\partial}{\partial y}\right) \\ &= \left(\frac{d}{ds} + 2\pi(-x \sin 2\pi s - y \cos 2\pi s) \frac{\partial}{\partial x} + 2\pi(-x \cos 2\pi s + y \sin 2\pi s) \frac{\partial}{\partial y}\right) \\ &\quad + \pi y \left(\cos 2\pi s \frac{\partial}{\partial x} - \sin 2\pi s \frac{\partial}{\partial y}\right) - \pi x \left(-\sin 2\pi s \frac{\partial}{\partial x} - \cos 2\pi s \frac{\partial}{\partial y}\right) \\ &= \frac{d}{ds} + \pi(-x \sin 2\pi s - y \cos 2\pi s) \frac{\partial}{\partial x} + \pi(-x \cos 2\pi s + y \sin 2\pi s) \frac{\partial}{\partial y} \\ &= \tilde{X}(\tilde{\omega}_3(x)). \end{aligned}$$

Thus, \tilde{X} is compatible with the involution $\tilde{\omega}_3$. This completes the proof of Lemma 4.2. □

We define the map $\Theta_{\tilde{X}} : \pi_r^{-1}(0) \rightarrow \pi_r^{-1}(0)$ as follows:

$$\begin{array}{ccc} \Theta_{\tilde{X}} : & \pi_r^{-1}(0) & \longrightarrow \pi_r^{-1}(0) \\ & \Downarrow & \Downarrow \\ & x & \longmapsto c_{\tilde{X},x}(1), \end{array}$$

where $c_{\tilde{X},x}$ is the integral curve of \tilde{X} starting at x . We identify the fiber $\pi_r^{-1}(0)$ with the surface Σ_{g-1} . Then the map $\Theta_{\tilde{X}}$ is contained in the centralizer $C(\iota)$ since \tilde{X} is compatible with $\tilde{\omega}_3$. Moreover, $\Theta_{\tilde{X}}$ is isotopic to the identity map. By Lemma 2.7, we can take an isotopy $\tilde{H}_t : \Sigma_{g-1} \rightarrow \Sigma_{g-1}$ which satisfies the following conditions:

- $\tilde{H}_0 = \Theta_{\tilde{X}}$;
- \tilde{H}_1 is the identity map;
- \tilde{H}_t is contained in the centralizer $C(\iota)$.

By using this isotopy, we obtain the following isomorphism as Σ_{g-1} -bundle:

$$\partial(M_h \cup M_r) \cong [0, 1] \times \Sigma_{g-1} / (1, x) \sim (0, x).$$

The involution $\tilde{\omega}_3$ acts on $[0, 1] \times \Sigma_{g-1} / (1, x) \sim (0, x)$ via the above isomorphism as follows:

$$\tilde{\omega}_3(s, x) = (s, \iota(x)).$$

Step 4: We define an involution ω_4 on $D^2 \times \Sigma_{g-1}$ as follows:

$$\omega_4(z, x) = (z, \iota(x)),$$

where $(z, x) \in D^2 \times \Sigma_{g-1}$. Then it is obvious that the following diagram commutes:

$$\begin{array}{ccc} [0, 1] \times \Sigma_{g-1} / \sim & \xrightarrow{\tilde{\omega}_3} & [0, 1] \times \Sigma_{g-1} / \sim \\ \Phi \downarrow & & \downarrow \Phi \\ \partial D^2 \times \Sigma_{g-1} & \xrightarrow{\omega_4} & \partial D^2 \times \Sigma_{g-1}, \end{array}$$

where Φ is the attaching map between M_r and $D^2 \times \Sigma_{g-1}$, which is given by $\Phi(s, x) = (\exp(2\pi\sqrt{-1}s), x)$. Hence we obtain an involution $\omega = \tilde{\omega}_3 \cup \omega_4$ on M .

We next look at the fixed point set of ω . ω is equal to $\text{id} \times \iota$ on $D^2 \times \Sigma_g$. So we obtain:

$$M^\omega \cup D^2 \times \Sigma_g = D^2 \times \{v_1, \dots, v_{2g+2}\},$$

where $v_1, \dots, v_{2g+2} \in \Sigma_g$ are the fixed points of ι . We remark that $M^\omega \cup D^2 \times \Sigma_g$ has the natural orientation derived from the orientation of D^2 .

ω acts on the 2-handle $h_i^2 = D^2 \times D^2$ as follows:

$$\omega(w_1, w_2) = \begin{cases} (\overline{w_1}, \overline{w_2}) & (c_i: \text{non-separating}), \\ (-w_1, -w_2) & (c_i: \text{separating}), \end{cases}$$

where $(w_1, w_2) \in D^2 \times D^2$. So the fixed point set $h_i^{2\omega}$ is equal to $(D^2 \cap \mathbb{R}) \times (D^2 \cap \mathbb{R})$ if c_i is non-separating and is equal to $\{(0, 0)\}$ if c_i is separating. Furthermore, if c_i is non-separating, we can give an orientation to $(D^2 \cap \mathbb{R}) \times (D^2 \cap \mathbb{R})$ which is compatible with the orientation of $D^2 \times \{v_1, \dots, v_{2g+2}\}$. Hence the fixed point set M_h^ω is the union of the oriented surfaces and the s points, where s is the number of Lefschetz singularities of f whose vanishing cycle is separating.

ω acts on the round 2-handle R^2 as follows:

$$\omega(s, z, t) = \begin{cases} (s, \bar{z}, -t) & (\text{if } H_1 \text{ preserves the two points } v_1 \text{ and } v_2), \\ (s, \exp(-2\pi\sqrt{-1}s)\bar{z}, -t) & (\text{otherwise}), \end{cases}$$

where $(s, z, t) \in R^2 = [0, 1] \times D^2 \times D^1 / \sim$. So we obtain:

$$R^{2\omega} = \begin{cases} [0, 1] \times (D^2 \cap \mathbb{R}) \times \{0\} / \sim & (\text{if } H_1 \text{ preserves the two points } v_1 \text{ and } v_2), \\ \{(s, z, 0) \in R^2 \mid z = r \exp(-\pi\sqrt{-1}s), r \in [-1, 1]\} & (\text{otherwise}). \end{cases}$$

Therefore, the fixed point set $R^{2\omega}$ is equal to the annulus or the Möbius band. Moreover, it is easy to show that the 2-dimensional part of the fixed point set $(M_h \cup M_r)^\omega$ does not admit any orientations even if $R^{2\omega}$ is equal to the annulus. So $(M_h \cup M_r)^\omega$ is the union of the unorientable surfaces and the s points.

ω is equal to $\text{id} \times \iota$ on $D^2 \times \Sigma_{g-1}$. So the fixed point set $(D^2 \times \Sigma_{g-1})^\omega$ is equal to $D^2 \times \{\tilde{v}_1, \dots, \tilde{v}_{2g}\}$, where $\{\tilde{v}_1, \dots, \tilde{v}_{2g}\} = \Sigma_{g-1}^\iota$. Eventually, M^ω is the union of the closed surfaces and the s points. The 2-dimensional part of M^ω is orientable if f has no indefinite fold singularities and is not orientable otherwise. This completes the proof of the statement in Theorem 1.1 on the fixed point set of ω .

We next extend the involution ω to the manifold $M_{\#s\mathbb{CP}^2}$. We assume that the curves c_{k_1}, \dots, c_{k_s} are separating. We construct the manifold $M_{\#s\mathbb{CP}^2}$ by blowing up M s times at $(0, 0) \in h_{k_i}^2$ ($i = 1, \dots, s$). In other word, $M_{\#s\mathbb{CP}^2}$ is decomposed as follows:

$$M_{\#s\mathbb{CP}^2} = D^2 \times \Sigma_g \cup (h_1^2 \amalg \dots \amalg h_n^2) \cup (\tilde{h}_{k_1} \amalg \dots \amalg \tilde{h}_{k_s}) \cup R^2 \cup D^2 \times \Sigma_{g-1},$$

where $\tilde{h}_{k_i} = \{((w_1, w_2), [l_1 : l_2]) \in D^2 \times D^2 \times \mathbb{CP}^1 \mid w_1 l_2 - w_2 l_1 = 0\} \cong h_{k_i}^2$. We define an involution $\bar{\omega}$ on $M_{\#s\mathbb{CP}^2}$ as follows:

$$\begin{aligned} \bar{\omega}(x) &= x & (x \in M_{\#s\mathbb{CP}^2} \setminus (\tilde{h}_{k_1} \amalg \dots \amalg \tilde{h}_{k_s})), \\ \bar{\omega}((w_1, w_2), [l_1 : l_2]) &= ((-w_1, -w_2), [l_1 : l_2]) & ((w_1, w_2), [l_1 : l_2]) \in \tilde{h}_{k_i}. \end{aligned}$$

It is obvious that $\bar{\omega}$ is an extension of ω . The fixed point set of $\bar{\omega}$ is the union of the 2-dimensional part of M^ω and s 2-spheres.

We next prove that $M_{\#s\mathbb{CP}^2}/\bar{\omega}$ is diffeomorphic to $S\sharp 2s\mathbb{CP}^2$, where S is an S^2 -bundle over S^2 . Since Σ_g/ι is diffeomorphic to S^2 , it is easy to see that $D^2 \times \Sigma_g/\bar{\omega}$ is diffeomorphic to $D^2 \times S^2$. So $M_{\#s\mathbb{CP}^2}$ is obtained by attaching $h_j/\bar{\omega}$ ($j \neq k_1, \dots, k_s$), $\tilde{h}_{k_i}/\bar{\omega}$, $R^2/\bar{\omega}$ and $D^2 \times \Sigma_{g-1}/\bar{\omega} \cong D^2 \times S^2$ to $D^2 \times S^2$.

Lemma 4.3. *Suppose that c_i is non-separating. Then,*

$$(D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2)/\bar{\omega} \cong D^2 \times S^2.$$

Proof of Lemma 4.3. If we identify $h_i^2 = D^2 \times D^2$ with D^4 , then $\bar{\omega}$ is equal to the covering transformation of the double covering $D^4 \rightarrow D^4$ branched at the unknotted 2-disk in D^4 . In particular, we obtain $h_i^2/\bar{\omega} \cong D^4$. Moreover, the attaching region of h_i^2 corresponds to the 3-disk in $\partial D^4 = \partial h_i^2/\bar{\omega}$. Denote by $\bar{\varphi}_i : h_i^2/\bar{\omega} \rightarrow \partial D^2 \times \Sigma_g/\bar{\omega}$ the embedding induced by φ_i . Then we obtain:

$$\begin{aligned} (D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2)/\bar{\omega} &\cong (D^2 \times \Sigma_g/\bar{\omega}) \cup_{\bar{\varphi}_i} h_i^2/\bar{\omega} \\ &\cong D^2 \times S^2 \natural D^4 \\ &\cong D^2 \times S^2. \end{aligned}$$

This completes the proof of Lemma 4.3. □

Lemma 4.4. *For each $i \in \{1, \dots, s\}$, $(D^2 \times \Sigma_g \cup_{\varphi_i} \tilde{h}_{k_i}^2)/\overline{\omega} \cong D^2 \times S^2 \# 2\overline{\mathbb{CP}^2}$.*

Proof of Lemma 4.4. By eliminating the corner of $D^2 \times D^2$, we identify $\tilde{h}_{k_i}^2$ with the following space:

$$H = \{((w_1, w_2), [l_1 : l_2]) \in D^4 \times \mathbb{CP}^1 \mid w_1 l_2 - w_2 l_1 = 0\}.$$

By the above identification, the attaching region of $\tilde{h}_{k_i}^2$ corresponds to the tubular neighborhood of the circle $\{((w_1, 0), [1 : 0]) \in \partial H \mid |w_1| = 1\}$ in ∂H . Let $p_2 : H \rightarrow \mathbb{CP}^1$ be the projection onto the second component. Then p_2 is the D^2 -bundle over the 2-sphere with Euler number -1 . We define $D_1, D_2 \subset \mathbb{CP}^1$ and local trivializations ψ_1 and ψ_2 of p_2 as follows:

$$\begin{aligned} D_1 &= \{[l_1 : l_2] \in \mathbb{CP}^1 \mid |l_1| \geq |l_2|\}, \\ D_2 &= \{[l_1 : l_2] \in \mathbb{CP}^1 \mid |l_2| \geq |l_1|\}, \\ \psi_1 : D^2 \times D^2 &\xrightarrow{\quad \quad \quad} p_2^{-1}(D_1) \\ &\quad \quad \quad \cup \quad \quad \quad \cup \\ (w_1, w_2) &\mapsto \left(\frac{w_2}{\sqrt{1 + |w_1|^2}}(1, w_1), [1, w_1] \right), \\ \psi_2 : D^2 \times D^2 &\xrightarrow{\quad \quad \quad} p_2^{-1}(D_2) \\ &\quad \quad \quad \cup \quad \quad \quad \cup \\ (w_1, w_2) &\mapsto \left(\frac{w_2}{\sqrt{1 + |w_1|^2}}(w_1, 1), [w_1, 1] \right). \end{aligned}$$

Denote $p_2^{-1}(D_1)$ and $p_2^{-1}(D_2)$ by H_1 and H_2 , respectively. We identify H_1 and H_2 with $D^2 \times D^2$ by the above trivializations. Then H can be identified with $D^2 \times D^2 \cup_{\Psi} D^2 \times D^2$, where $\Psi = \psi_1^{-1} \circ \psi_2 : (w_1, w_2) \mapsto (\frac{1}{w_1}, w_1 w_2)$. We remark that the attaching region of H corresponds to $\partial D^2 \times D^2 \subset \partial H_1$.

We define $\tilde{H} = \tilde{H}_1 \cup_{\tilde{\Psi}} \tilde{H}_2$, where $\tilde{H}_i = D^2 \times D^2$ ($i = 1, 2$) and $\tilde{\Psi} : \partial D^2 \times D^2 \rightarrow \partial D^2 \times D^2$ is a diffeomorphism defined as follows:

$$\tilde{\Psi}(w_1, w_2) = (\frac{1}{w_1}, w_1^2 w_2).$$

Then we can define $\mathcal{P} : H \rightarrow \tilde{H}$ as follows:

$$\mathcal{P}(w_1, w_2) = \begin{cases} (w_1, w_2^2) \in \tilde{H}_1 & ((w_1, w_2) \in H_1), \\ (w_1, w_2^2) \in \tilde{H}_2 & ((w_1, w_2) \in H_2). \end{cases}$$

The map \mathcal{P} is a double branched covering branched at the 0-section of \tilde{H} as a D^2 -bundle. Moreover, $\tilde{\omega}$ is the non-trivial covering transformation of \mathcal{P} . So we obtain $H/\tilde{\omega} \cong \tilde{H}$.

Since the attaching region of H is mapped by \mathcal{P} to $D^2 \times \partial D^2 \subset \partial \tilde{H}_1$, we can regard \tilde{H}_1 and \tilde{H}_2 as 2-handles. Thus, $(D^2 \times \Sigma_g \cup_{\varphi_i} \tilde{h}_{k_i}^2)/\overline{\omega}$ is obtained by attaching the 2-handles \tilde{H}_1 and \tilde{H}_2 to $D^2 \times S^2$. To prove the statement, we look at the attaching maps of \tilde{H}_1 and \tilde{H}_2 .

We take an identification $\nu c_{k_i} \cong J \times S^1$ as we take in Step 2 of the construction of ω . Then the attaching map φ_{k_i} of the 2-handle $h_{k_i}^2$ satisfies $\varphi_{k_i}(w_1, w_2) = (\varepsilon_2 w_2 w_1, w_1)$. Since H is obtained by

eliminating the corner of $\tilde{h}_{k_i}^2$, The attaching map of H_1 is described as follows:

$$\begin{array}{ccc} \Phi: \partial H_1 \supset D^2 \times \partial D^2 & \longrightarrow & J \times S^1 \\ \Psi \quad \quad \quad \Psi & & \\ (w_1, w_2) & \longmapsto & (\varepsilon_2 w_2^2 w_1, w_2). \end{array}$$

For an element $(z_1, z_2) \in J \times S^1$, $\overline{\omega}(z_1, z_2) = (z_1, -z_2)$. So we obtain $J \times S^1 / \overline{\omega} \cong J \times S^1$ and the quotient map $\overline{\omega}: J \times S^1 \rightarrow J \times S^1 / \overline{\omega} \cong J \times S^1$ satisfies $\overline{\omega}(z_1, z_2) = (z_1, z_2^2)$. Thus, the attaching map $\tilde{\Phi}: D^2 \times \partial D^2 \rightarrow J \times S^1$ of \tilde{H}_1 satisfies $\tilde{\Phi}(w_1, w_2) = (\varepsilon_2 w_2 w_1, w_2)$. It is easy to see that the attaching circle of \tilde{H}_1 is equal to the circle $c_{k_i} / \overline{\omega}$. Moreover, the framing of $\tilde{\Phi}$ is -1 relative to the framing along $\{*\} \times S^2 \subset \partial D^2 \times S^2$.

By the definition of $\tilde{\Psi}$, the attaching circle of \tilde{H}_2 is equal to the belt circle of \tilde{H}_1 , which is isotopic to the meridian of the attaching circle of \tilde{H}_1 . In particular, there exists the natural framing of the attaching circle of \tilde{H}_2 which is represented by the meridian of the attaching circle of \tilde{H}_1 parallel to the attaching circle of \tilde{H}_2 . Since the Euler number of \tilde{H} as a D^2 -bundle is equal to -2 , the framing of the attaching map $\tilde{\Psi}$ is equal to -2 relative to the natural framing. Therefore, we can draw a Kirby diagram of $(D^2 \times \Sigma_g \cup_{\varphi_i} \tilde{h}_{k_i}^2) / \overline{\omega}$ as shown in Figure 7. It is obvious that this manifold is diffeomorphic to $D^2 \times S^2 \# 2\mathbb{CP}^2$ and this completes the proof of Lemma 4.4.

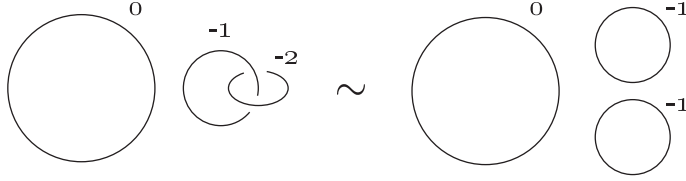


FIGURE 7. the -1 -framed knot describes \tilde{H}_1 , while the -2 -framed knot describes \tilde{H}_2 .

□

By applying the arguments in Lemma 4.3 and 4.4 successively, we obtain $M_h \# s\overline{\mathbb{CP}^2} / \overline{\omega} \cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$.

Lemma 4.5. $((M_h \cup M_r) \# s\overline{\mathbb{CP}^2}) / \overline{\omega} \cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$.

Proof of Lemma 4.5. We can decompose R^2 into two components as follows:

$$R^2 = \left[0, \frac{1}{2}\right] \times D^2 \times D^1 \cup \left[\frac{1}{2}, 1\right] \times D^2 \times D^1.$$

Denote $[0, \frac{1}{2}] \times D^2 \times D^1$ and $[\frac{1}{2}, 1] \times D^2 \times D^1$ by R_1 and R_2 , respectively. It is easy to see that $R_i / \overline{\omega}$ is diffeomorphic to D^4 and R_i is the double covering of $D^4 \cong R_i / \overline{\omega}$ branched at the unknotted 2-disk.

The attaching region of R_1 is equal to $[0, \frac{1}{2}] \times \partial D^2 \times D^1$. The quotient $[0, \frac{1}{2}] \times \partial D^2 \times D^1 / \overline{\omega}$ is a 3-ball in $\partial D^4 \cong \partial R_1$. Thus, we obtain:

$$\begin{aligned} (M_h \cup R_1) / \overline{\omega} &\cong M_h / \overline{\omega} \cup R_1 / \overline{\omega} \\ &\cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2} \# D^4 \\ &\cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}. \end{aligned}$$

The attaching region of R_2 is equal to $[\frac{1}{2}, 1] \times \partial D^2 \times D^1 \cup \{\frac{1}{2}, 1\} \times D^2 \times D^1$. The quotient $[\frac{1}{2}, 1] \times \partial D^2 \times D^1 / \overline{\omega}$ is a 3-ball D_0 in $\partial D^4 \cong \partial R_2$, while $\{\frac{1}{2}, 1\} \times D^2 \times D^1 / \overline{\omega}$ is a disjoint union of two 3-balls $D_1 \amalg D_2$ in ∂D^4 . Both of the intersections $D_0 \cap D_1$ and $D_0 \cap D_2$ are 2-disks in ∂D_0 . Eventually, the attaching region of R_2 is a 3-ball in ∂D^4 . So we obtain $(M_h \cup R_1 \cup R_2) / \overline{\omega} \cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$. This completes the proof of Lemma 4.5. \square

It is easy to see that $D^2 \times \Sigma_{g-1} / \overline{\omega}$ is diffeomorphic to $D^2 \times S^2$ and attached to $(M_h \cup M_r) / \overline{\omega}$ so that the following diagram commutes:

$$\begin{array}{ccccc} (M_h \cup M_r) / \overline{\omega} \supset & S^1 \times S^2 & \longrightarrow & \partial D^2 \times S^2 & \subset D^2 \times \Sigma_{g-1} / \overline{\omega} \\ & \downarrow & & \downarrow & \\ & S^1 & \longrightarrow & \partial D^2, & \end{array}$$

where the upper horizontal arrow in the diagram represents the attaching map, the lower horizontal arrow represents the identity map and vertical arrows represent the projection onto the first component (In other word, the attaching map is a bundle map as a S^2 -bundle over S^1). In particular, we obtain:

$$M \# s\overline{\mathbb{CP}^2} / \overline{\omega} \cong S \# 2s\overline{\mathbb{CP}^2}.$$

It is obvious that the quotient map $/\overline{\omega} : M \# s\overline{\mathbb{CP}^2} \rightarrow S \# 2s\overline{\mathbb{CP}^2}$ is a double branched covering. Thus, we complete the proof of the statement (i) in Theorem 1.1. \square

Proof of (ii) in Theorem 1.1. Let $F_h \subset M$ be a regular fiber in the higher side of f . It is easy to see that F_h represents the same rational homology class of M as the one F represents. Let $\omega : M \rightarrow M$ be the involution constructed in the proof of (i) in Theorem 1.1. If f has no indefinite fold singularities, that is, f is a Lefschetz fibration, then the 2-dimensional part of the fixed point set M^ω of the involution ω is an orientable surface and the algebraic intersection number between this part and F_h is equal to $2g + 2$, especially is non-zero. So the statement (ii) in Theorem 1.1 holds.

Suppose that f has indefinite fold singularities. We first prove that F_h represents a non-trivial rational homology class of $M_h \cup M_r$. To prove this, we construct an element \mathcal{S} in the group $H_2(M_h \cup M_r, \partial(M_h \cup M_r); \mathbb{Q})$ such that $[F_h] \cdot \mathcal{S} \neq 0$. Let \tilde{S} be the intersection between the 2-dimensional part of M^ω and M_h , which is the union of compact oriented surfaces. We use the notations H_1 , c , v_1 , v_2 and R^2 as we use in the proof of (i) in Theorem 1.1.

Case 1: If H_1 preserves the orientation of c and two points v_1 and v_2 , then R^2 is untwisted and $\tilde{S} \cap R^2 = \{(s, \pm 1, 0) \in R^2 \mid s \in [0, 1]\}$ is a disjoint union of two circles. We define four annuli A_1 , A_2 , A_3 and A_4 as follows:

$$\begin{aligned} A_1 &= \{(s, t, 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\ A_2 &= \{(s, t, 0) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}, \\ A_3 &= \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\ A_4 &= \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}. \end{aligned}$$

Then $S = \tilde{S} \cup A_1 \cup A_2 \cup A_3 \cup A_4$ represents the homology class of the pair $(M_h \cup M_r, \partial(M_h \cup M_r))$ after giving suitable orientations to the annuli A_1 , A_2 , A_3 and A_4 . We denote this class by \mathcal{S} , then the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g + 2$, especially is non-zero.

Case 2: If H_1 preserves the orientation of c but does not preserve two points v_1 and v_2 , then R^2 is untwisted and $\tilde{S} \cap R^2 = \{(s, \pm \exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1]\}$ is a circle. We define three annuli A_5 , A_6 and A_7 as follows:

$$\begin{aligned} A_5 &= \{(s, t \exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\} \\ &\quad \cup \{(s, -t \exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\ A_6 &= \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\ A_7 &= \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}. \end{aligned}$$

Then $S = \tilde{S} \cup A_5 \cup A_6 \cup A_7$ represents the homology class of the pair $(M_h \cup M_r, \partial(M_h \cup M_r))$ after giving suitable orientations to the annuli A_5 , A_6 and A_7 . We denote this class by \mathcal{S} , then the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g + 2$, especially is non-zero.

Case 3: If H_1 does not preserve the orientation of c but preserves two points v_1 and v_2 , then R^2 is twisted and $\tilde{S} \cap R^2 = \{(s, \pm 1, 0) \in R^2 \mid s \in [0, 1]\}$ is a disjoint union of two circles. We define three annuli A_8 , A_9 and A_{10} as follows:

$$\begin{aligned} A_8 &= \{(s, t, 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\ A_9 &= \{(s, t, 0) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}, \\ A_{10} &= \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [0, 1]\} \\ &\quad \cup \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}. \end{aligned}$$

Then $S = \tilde{S} \cup A_8 \cup A_9 \cup A_{10}$ represents the homology class of the pair $(M_h \cup M_r, \partial(M_h \cup M_r))$ after giving suitable orientations to the annuli A_8 , A_9 and A_{10} . We denote this class by \mathcal{S} , then the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g + 2$, especially is non-zero.

Case 4: If H_1 preserves neither the orientation of c nor two points v_1 and v_2 , then R^2 is twisted and $\tilde{S} \cap R^2 = \{(s, \pm \exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1]\}$ is a circle. We define two annuli A_{11} and A_{12} as follows:

$$\begin{aligned} A_{11} &= \{(s, t \exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\} \\ &\quad \cup \{(s, -t \exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\ A_{12} &= \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\ &\quad \cup \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}. \end{aligned}$$

Then $S = \tilde{S} \cup A_{11} \cup A_{12}$ represents the homology class of the pair $(M_h \cup M_r, \partial(M_h \cup M_r))$ after giving suitable orientations to the annuli A_{11} and A_{12} . We denote this class by \mathcal{S} , then the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g + 2$, especially is non-zero.

Eventually, we can construct the element \mathcal{S} satisfying the desired condition in any cases. So we have proved $[F_h] \neq 0$ in $H_2(M_h \cup M_r; \mathbb{Q})$.

We are now ready to prove the statement (ii) in Theorem 1.1. There exists the following exact sequence which is the part of the Meyer-Vietoris exact sequence:

$$H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \xrightarrow{i_1 \oplus i_2} H_2(M_h \cup M_r; \mathbb{Q}) \oplus H_2(D^2 \times \Sigma_{g-1}; \mathbb{Q}) \xrightarrow{j_1 - j_2} H_2(M; \mathbb{Q}).$$

Suppose that $(j_1 - j_2)([F_h], 0) = [F_h] = 0$. Then there exists an element $\mu \in H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q})$ such that $(i_1 \oplus i_2)(\mu) = ([F_h], 0)$. By a Künneth formula, we obtain the following isomorphism:

$$H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \cong H_2(\Sigma_{g-1}; \mathbb{Q}) \oplus (H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})).$$

Since the map $i_2 : H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \rightarrow H_2(D^2 \times \Sigma_{g-1}; \mathbb{Q}) \cong H_2(\Sigma_{g-1}; \mathbb{Q})$ is regarded as the projection onto the first component via the above isomorphism, μ is an element in $H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})$. The involution ω acts on the component $H_2(\Sigma_{g-1}; \mathbb{Q})$ trivially and on the component $H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})$ by multiplying -1 . So we obtain:

$$\omega_*(\mu) = -\mu.$$

$i_1 \circ \omega_*$ is equal to $\omega_* \circ i_1$ since i_1 is induced by the inclusion map. Thus, we obtain:

$$\begin{aligned} [F_h] &= \omega_*([F_h]) \\ &= \omega_* \circ i_1(\mu) \\ &= i_1 \circ \omega_*(\mu) \\ &= i_1 \circ (-\mu) = -[F_h]. \end{aligned}$$

This means that $2[F_h] = 0$ in $H_2(M_h \cup M_r; \mathbb{Q})$. This contradicts $[F_h] \neq 0$. Therefore, we obtain $[F_h] \neq 0$ in $H_2(M; \mathbb{Q})$ and this completes the proof of the statement. \square

Remark 4.6. By the argument similar to that in the proof of Theorem 1.1, we can generalize Theorem 1.1 to directed BLFs as follows:

Theorem 4.7. *Let $f : M \rightarrow S^2$ be a hyperelliptic directed BLF. Suppose that the genus of every fiber component of f is greater than or equal to 2.*

- (i) *Let s_1 be the number of Lefschetz singularities of f whose vanishing cycles are separating. We define s_2 as follows:*

$$s_2 = \max\{s \in \mathbb{N} \mid f^{-1}(x) \text{ has } s \text{ components. } x \in S^2\}.$$

Then there exists an involution

$$\omega : M \rightarrow M$$

such that the fixed point set of ω is the union of (possibly nonorientable) surfaces and s_1 isolated points. Moreover, ω can be extended to an involution

$$\overline{\omega} : M\sharp_{s_1}\overline{\mathbb{CP}^2} \rightarrow M\sharp_{s_1}\overline{\mathbb{CP}^2}$$

such that $M\sharp_{s_1}\overline{\mathbb{CP}^2}/\overline{\omega}$ is diffeomorphic to $\sharp_{s_2}S\sharp_{2s_1}\overline{\mathbb{CP}^2}$, where S is S^2 -bundle over S^2 , and the quotient map

$$/\overline{\omega} : M\sharp_{s_1}\overline{\mathbb{CP}^2} \rightarrow M\sharp_{s_1}\overline{\mathbb{CP}^2}/\overline{\omega} \cong \sharp_{s_2}S\sharp_{2s_1}\overline{\mathbb{CP}^2}$$

is the double branched covering.

- (ii) *Let $F \in M$ be a regular fiber of f . Then F represents a non-trivial rational homology class of M .*

5. A GENUS-2 SBLF STRUCTURE ON $\#n\mathbb{CP}^2$ FOR $n \geq 0$

In this section, we prove Theorem 1.5. Let $\tilde{c}_{1,1}, \tilde{c}_{1,2}, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \tilde{c}_5 \subset \Sigma_{2,1}$ be simple closed curves described as in Figure 8.

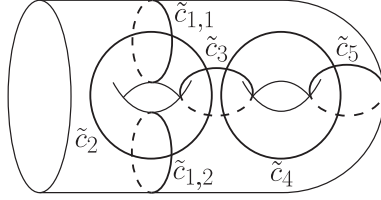


FIGURE 8.

We denote by $\tilde{t}_{1,1}, \tilde{t}_{1,2}, \tilde{t}_2, \tilde{t}_3, \tilde{t}_4, \tilde{t}_5 \in \mathcal{M}_{2,1}$ right-handed Dehn twists along the simple closed curves $\tilde{c}_{1,1}, \tilde{c}_{1,2}, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \tilde{c}_5$, respectively. We define the simple closed curves $\alpha_i, \beta_j, \gamma_k \subset \Sigma_{2,1}$ as follows:

$$\tilde{\alpha}_i = (\tilde{t}_5^2 \cdot \tilde{t}_4^{-i+1})(\tilde{c}_4) \quad (i \in \mathbb{Z}),$$

$$\tilde{\beta}_j = \tilde{t}_4^{-j}(\tilde{c}_5) \quad (j \in \mathbb{Z}),$$

$$\tilde{\gamma}_1 = (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4 \cdot \tilde{t}_5^2)(\tilde{c}_{1,1}),$$

$$\tilde{\gamma}_2 = (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4)(\tilde{c}_{1,1}).$$

These curves are described as in Figure 9.

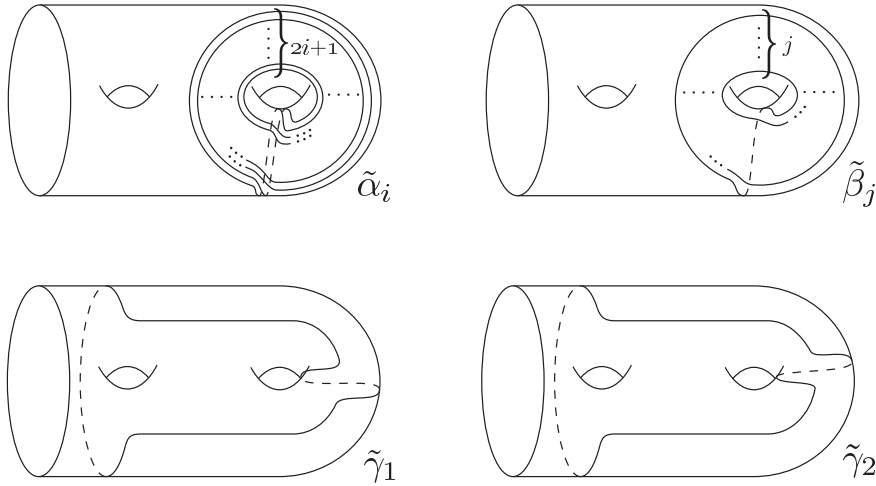


FIGURE 9.

We also define elements $\tilde{\xi}, \tilde{\iota}_2 \in \mathcal{M}_{2,1}$ as follows:

$$\tilde{\xi} = (\tilde{t}_4 \cdot \tilde{t}_5)^3,$$

$$\tilde{\iota}_2 = (\tilde{t}_5 \cdot \tilde{t}_4 \cdot \tilde{t}_3 \cdot \tilde{t}_2)^5$$

$$= \tilde{t}_{1,1} \cdot \tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4 \cdot \tilde{t}_3 \cdot \tilde{t}_2 \cdot \tilde{t}_{1,1}.$$

We remark that the element $\tilde{\iota}_2$ is a lift of the hyperelliptic involution $\iota_2 \in \mathcal{M}_2$ and that the element $\tilde{\xi}$ is a lift of the element $\xi \in \mathcal{M}_2$ which is represented by the element described as in Figure 10.

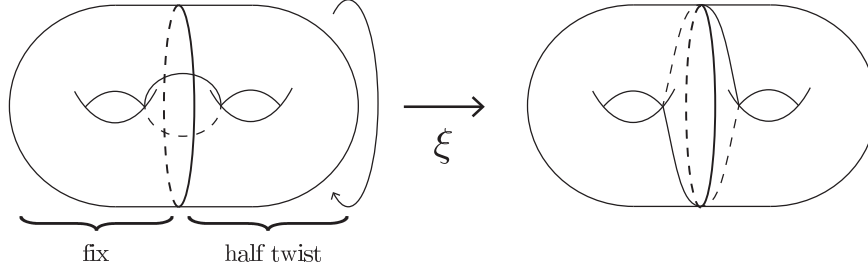


FIGURE 10.

We can easily obtain the following formulas:

$$\begin{aligned}\tilde{t}_2 \cdot \tilde{t}_j &= \tilde{t}_j \cdot \tilde{t}_2 & (j = 2, 3, 4, 5), \\ \tilde{t}_2 \cdot \tilde{t}_{1,1} &= \tilde{t}_{1,2} \cdot \tilde{t}_2.\end{aligned}$$

Lemma 5.1. $t_{\tilde{\gamma}_1} \cdot t_{\tilde{\gamma}_2} = \tilde{t}_2 \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-2} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1} \in \text{Ker}(\Phi_{\tilde{c}_5} : \mathcal{M}_{2,1}(\tilde{c}_5) \rightarrow \mathcal{M}_{1,1})$, where $\mathcal{M}_{2,1}(\tilde{c}_5)$ is the subgroup of $\mathcal{M}_{2,1}$ which is defined as follows:

$$\mathcal{M}_{2,1}(\tilde{c}_5) = \{[T] \in \mathcal{M}_{2,1} \mid T(\tilde{c}_5) = \tilde{c}_5\}.$$

Proof. We first prove the equation by direct calculation. By the definitions of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, we obtain:

$$\begin{aligned}t_{\tilde{\gamma}_1} &= (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4 \cdot \tilde{t}_5^2)^{-1} \cdot \tilde{t}_{1,1} \cdot (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4 \cdot \tilde{t}_5^2), \\ t_{\tilde{\gamma}_2} &= (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4)^{-1} \cdot \tilde{t}_{1,1} \cdot (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4).\end{aligned}$$

So we can calculate $t_{\tilde{\gamma}_1} \cdot t_{\tilde{\gamma}_2}$ as follows:

$$\begin{aligned}t_{\tilde{\gamma}_1} \cdot t_{\tilde{\gamma}_2} &= (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4 \cdot \tilde{t}_5^2)^{-1} \cdot \tilde{t}_{1,1} \cdot (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4 \cdot \tilde{t}_5^2) \cdot (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4)^{-1} \cdot \tilde{t}_{1,1} \cdot (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4) \\ &= \tilde{t}_5^{-2} \cdot \tilde{t}_4^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot (\tilde{t}_{1,1} \cdot \tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4 \cdot \tilde{t}_5^2) \cdot \tilde{t}_4^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1} \cdot \tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4 \\ &= \tilde{t}_5^{-2} \cdot \tilde{t}_4^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot (\tilde{t}_2 \cdot \tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_4^{-1}) \cdot \tilde{t}_4^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1} \cdot \tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4 \\ &= \tilde{t}_2 \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_4^{-1} \cdot \tilde{t}_3^{-1} \cdot (\tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1}) \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_4^{-2} \cdot \tilde{t}_3^{-1} \cdot (\tilde{t}_2^{-1} \cdot \tilde{t}_{1,1} \cdot \tilde{t}_2) \cdot \tilde{t}_3 \cdot \tilde{t}_4 \\ &= \tilde{t}_2 \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_4^{-1} \cdot \tilde{t}_3^{-1} \cdot (\tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1}) \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_4^{-2} \cdot \tilde{t}_3^{-1} \cdot (\tilde{t}_{1,1} \cdot \tilde{t}_2 \cdot \tilde{t}_{1,1}^{-1}) \cdot \tilde{t}_3 \cdot \tilde{t}_4 \\ &= \tilde{t}_2 \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_{1,1}^{-1} \cdot \tilde{t}_4^{-1} \cdot (\tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1}) \cdot \tilde{t}_4^{-2} \cdot (\tilde{t}_3^{-1} \cdot \tilde{t}_2 \cdot \tilde{t}_3) \cdot \tilde{t}_4 \cdot \tilde{t}_{1,1}^{-1} \\ &= \tilde{t}_2 \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_{1,1}^{-1} \cdot \tilde{t}_4^{-1} \cdot (\tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1}) \cdot \tilde{t}_4^{-2} \cdot (\tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_2^{-1}) \cdot \tilde{t}_4 \cdot \tilde{t}_{1,1}^{-1} \\ &= \tilde{t}_2 \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot (\tilde{t}_4^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_4^{-1}) \cdot (\tilde{t}_4^{-1} \cdot \tilde{t}_3 \cdot \tilde{t}_4) \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1} \\ &= \tilde{t}_2 \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot (\tilde{t}_3^{-1} \cdot \tilde{t}_4^{-1} \cdot \tilde{t}_3^{-1}) \cdot (\tilde{t}_3 \cdot \tilde{t}_4 \cdot \tilde{t}_3^{-1}) \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1} \\ &= \tilde{t}_2 \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1}.\end{aligned}$$

We next prove that the element $\tilde{t}_2 \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1}$ is contained in the kernel of $\Phi_{\tilde{c}_5} : \mathcal{M}_{2,1}(\tilde{c}_5) \rightarrow \mathcal{M}_{1,1}$. The elements \tilde{t}_2, \tilde{t}_5 and $\tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1}$ are contained in the

group $\mathcal{M}_{2,1}(\tilde{c}_5)$. It is obvious that $\Phi_{\tilde{c}_5}(\tilde{t}_5) = 1$. We can calculate the product $\tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4$ as follows:

$$\begin{aligned}
 \tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4 &= \tilde{t}_5^{-1} \cdot (\tilde{t}_5 \cdot \tilde{t}_4 \cdot \tilde{t}_5) \cdot (\tilde{t}_5 \cdot \tilde{t}_4 \cdot \tilde{t}_5) \cdot \tilde{t}_5^{-1} \\
 &= \tilde{t}_5^{-1} \cdot (\tilde{t}_4 \cdot \tilde{t}_5 \cdot \tilde{t}_4) \cdot (\tilde{t}_5 \cdot \tilde{t}_4 \cdot \tilde{t}_5) \cdot \tilde{t}_5^{-1} \\
 &= \tilde{t}_5^{-1} \cdot \tilde{\xi} \cdot \tilde{t}_5^{-1}
 \end{aligned}
 \tag{5.1}$$

Since $\tilde{\xi} \in \mathcal{M}_{2,1}(\tilde{c}_5)$ and $\Phi_{\tilde{c}_5}(\tilde{\xi}) = 1$, we obtain:

$$\begin{aligned}
 &\Phi_{\tilde{c}_5}(\tilde{t}_2 \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1}) \\
 &= \Phi_{\tilde{c}_5}(\tilde{t}_2) \cdot \Phi_{\tilde{c}_5}(\tilde{t}_5^{-2}) \cdot \Phi_{\tilde{c}_5}(\tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1}) \\
 &= \Phi_{\tilde{c}_5}(\tilde{t}_{1,1} \cdot \tilde{t}_2 \cdot \tilde{t}_3 \cdot \tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4 \cdot \tilde{t}_3 \cdot \tilde{t}_2 \cdot \tilde{t}_{1,1}) \cdot \Phi_{\tilde{c}_5}(\tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1}) \\
 &= \Phi_{\tilde{c}_5}(\tilde{t}_{1,1} \cdot \tilde{t}_2 \cdot \tilde{t}_3) \cdot \Phi_{\tilde{c}_5}(\tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4) \cdot \Phi_{\tilde{c}_5}(\tilde{t}_3 \cdot \tilde{t}_2 \cdot \tilde{t}_{1,1}) \cdot \Phi_{\tilde{c}_5}(\tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1}) \\
 &= \Phi_{\tilde{c}_5}(\tilde{t}_{1,1} \cdot \tilde{t}_2 \cdot \tilde{t}_3) \cdot \Phi_{\tilde{c}_5}(\tilde{t}_3 \cdot \tilde{t}_2 \cdot \tilde{t}_{1,1}) \cdot \Phi_{\tilde{c}_5}(\tilde{t}_{1,1}^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_3^{-1} \cdot \tilde{t}_2^{-1} \cdot \tilde{t}_{1,1}^{-1}) \\
 &= 1.
 \end{aligned}$$

This completes the proof of Lemma 5.1. □

Lemma 5.2. $t_{\tilde{\beta}_1} \cdot t_{\tilde{\beta}_{-1}} = \tilde{\xi} \cdot \tilde{t}_5^{-4} \in \text{Ker } \Phi_{\tilde{c}_5}$.

Proof. By the definitions of the curves $\tilde{\beta}_1, \tilde{\beta}_2$, we obtain:

$$\begin{aligned}
 t_{\tilde{\beta}_1} &= \tilde{t}_4 \cdot \tilde{t}_5 \cdot \tilde{t}_4^{-1}, \\
 t_{\tilde{\beta}_{-1}} &= \tilde{t}_4^{-1} \cdot \tilde{t}_5 \cdot \tilde{t}_4.
 \end{aligned}$$

So we can calculate $t_{\tilde{\beta}_1} \cdot t_{\tilde{\beta}_{-1}}$ as follows:

$$\begin{aligned}
 t_{\tilde{\beta}_1} \cdot t_{\tilde{\beta}_{-1}} &= (\tilde{t}_4 \cdot \tilde{t}_5 \cdot \tilde{t}_4^{-1}) \cdot (\tilde{t}_4^{-1} \cdot \tilde{t}_5 \cdot \tilde{t}_4) \\
 &= \tilde{t}_5^{-1} \cdot \tilde{t}_4 \cdot \tilde{t}_5 \cdot \tilde{t}_5 \cdot \tilde{t}_4 \cdot \tilde{t}_5^{-1} \\
 &= \tilde{t}_5^{-2} \cdot \tilde{\xi} \cdot \tilde{t}_5^{-2} \quad (\text{by (5.1)}) \\
 &= \tilde{\xi} \cdot \tilde{t}_5^{-4}.
 \end{aligned}$$

Since $\Phi_{\tilde{c}_5}(\tilde{t}_5) = 1$ and $\Phi_{\tilde{c}_5}(\tilde{\xi}) = 1$, the element $\tilde{\xi} \cdot \tilde{t}_5^{-4}$ is contained in the kernel of $\Phi_{\tilde{c}_5}$. □

Lemma 5.3. For $s \geq 3$,

$$t_{\tilde{\alpha}_1} \cdot t_{\tilde{\alpha}_2} \cdots t_{\tilde{\alpha}_{s-2}} \cdot t_{\tilde{\beta}_{s-1}} \cdot t_{\tilde{\beta}_{-1}} = \tilde{\xi}^{s-1} \cdot \tilde{t}_5^{-5s+6} \in \text{Ker } \Phi_{\tilde{c}_5}.$$

Proof. We denote by $P_s \in \mathcal{M}_{2,1}$ the left side of the equation in the statement of Lemma 5.3. Then the following equations hold:

$$\begin{aligned}
 P_3 &= t_{\tilde{\beta}_1} \cdot t_{\tilde{\beta}_{-1}} \cdot (t_{\tilde{\beta}_{-1}}^{-1} \cdot t_{\tilde{\beta}_1}^{-1} \cdot t_{\tilde{\alpha}_1} \cdot t_{\tilde{\beta}_2} \cdot t_{\tilde{\beta}_{-1}}), \\
 P_s &= P_{s-1} \cdot (t_{\tilde{\beta}_{-1}}^{-1} \cdot t_{\tilde{\beta}_{s-2}}^{-1} \cdot t_{\tilde{\alpha}_{s-2}} \cdot t_{\tilde{\beta}_{s-1}} \cdot t_{\tilde{\beta}_{-1}}) \quad (s \geq 4).
 \end{aligned}$$

So, by Lemma 5.2, it is sufficient to prove the following equation:

$$t_{\tilde{\beta}_{-1}}^{-1} \cdot t_{\tilde{\beta}_{s-2}}^{-1} \cdot t_{\tilde{\alpha}_{s-2}} \cdot t_{\tilde{\beta}_{s-1}} \cdot t_{\tilde{\beta}_{-1}} = \tilde{\xi} \cdot \tilde{t}_5^{-5} \quad (s \geq 3).$$

We prove this equation by direct calculation. By the definitions of the curves α_i and β_j , we obtain:

$$\begin{aligned} t_{\tilde{\alpha}_i} &= (\tilde{t}_5^2 \cdot \tilde{t}_4^{-i+1})^{-1} \cdot \tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4^{-i+1}, \\ t_{\tilde{\beta}_j} &= \tilde{t}_4^j \cdot \tilde{t}_5 \cdot \tilde{t}_4^{-j}. \end{aligned}$$

So we can calculate $t_{\tilde{\beta}_{-1}}^{-1} \cdot t_{\tilde{\beta}_{s-2}}^{-1} \cdot t_{\tilde{\alpha}_{s-2}} \cdot t_{\tilde{\beta}_{s-1}} \cdot t_{\tilde{\beta}_{-1}}$ as follows:

$$\begin{aligned} & t_{\tilde{\beta}_{-1}}^{-1} \cdot t_{\tilde{\beta}_{s-2}}^{-1} \cdot t_{\tilde{\alpha}_{s-2}} \cdot t_{\tilde{\beta}_{s-1}} \cdot t_{\tilde{\beta}_{-1}} \\ &= (\tilde{t}_4^{-1} \cdot \tilde{t}_5^{-1} \cdot \tilde{t}_4) \cdot (\tilde{t}_4^{s-2} \cdot \tilde{t}_5^{-1} \cdot \tilde{t}_4^{-s+2}) \cdot ((\tilde{t}_5^2 \cdot \tilde{t}_4^{-s+3})^{-1} \cdot \tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4^{-s+3}) \cdot (\tilde{t}_4^{s-1} \cdot \tilde{t}_5 \cdot \tilde{t}_4^{-s+1}) \cdot (\tilde{t}_4^{-1} \cdot \tilde{t}_5 \cdot \tilde{t}_4) \\ &= \tilde{t}_4^{-1} \cdot \tilde{t}_5^{-1} \cdot \tilde{t}_4^{s-1} \cdot \tilde{t}_5^{-1} \cdot \tilde{t}_4^{-s+2} \cdot \tilde{t}_4^{-3} \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4^2 \cdot \tilde{t}_5 \cdot \tilde{t}_4^{-s} \cdot \tilde{t}_5 \cdot \tilde{t}_4 \\ &= (\tilde{t}_4^{-1} \cdot \tilde{t}_5^{-1} \cdot \tilde{t}_4^{s-1}) \cdot \tilde{t}_5^{-1} \cdot \tilde{t}_4^{-1} \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4^2 \cdot \tilde{t}_5 \cdot (\tilde{t}_4^{-s} \cdot \tilde{t}_5 \cdot \tilde{t}_4) \\ &= (\tilde{t}_5^{-1} \cdot \tilde{t}_4^{-1} \cdot \tilde{t}_5^{-1}) \cdot \tilde{t}_5^{-1} \cdot \tilde{t}_4^{-1} \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4^2 \cdot \tilde{t}_5 \cdot (\tilde{t}_5 \cdot \tilde{t}_4 \cdot \tilde{t}_5^{-s}) \\ &= \tilde{t}_5^{s-1} \cdot (\tilde{t}_4^{-1} \cdot \tilde{t}_5^{-2} \cdot \tilde{t}_4^{-1}) \cdot \tilde{t}_5^{-2} \cdot (\tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4) \cdot (\tilde{t}_4 \cdot \tilde{t}_5^2 \cdot \tilde{t}_4) \cdot \tilde{t}_5^{-s} \\ &= \tilde{t}_5^{s-1} \cdot (\tilde{t}_5^2 \cdot \tilde{\xi}^{-1}) \cdot \tilde{t}_5^{-2} \cdot (\tilde{t}_5^{-2} \cdot \tilde{\xi}) \cdot (\tilde{t}_5^{-2} \cdot \tilde{\xi}) \cdot \tilde{t}_5^{-s} \\ &= \tilde{\xi}^{s-1} \cdot \tilde{t}_5^{-5s+6}. \end{aligned}$$

This completes the proof of Lemma 5.3. □

For each $s \geq 2$, we define a sequence W_s of elements of the mapping class group \mathcal{M}_2 as follows:

$$W_s = \begin{cases} (t_{\beta_1}, t_{\beta_{-1}}, t_{\gamma_1}, t_{\gamma_2}) & (s = 2), \\ (t_{\alpha_1}, t_{\alpha_2}, \dots, t_{\alpha_{s-2}}, t_{\beta_{s-1}}, t_{\beta_{-1}}, t_{\gamma_1}, t_{\gamma_2}) & (s \geq 3), \end{cases}$$

where $\alpha_i, \beta_j, \gamma_k \subset \Sigma_2$ are the images of $\tilde{\alpha}_i, \tilde{\beta}_j, \tilde{\gamma}_k \subset \Sigma_{2,1}$ by the natural inclusion $\Sigma_{2,1} \hookrightarrow \Sigma_2$. By Lemma 5.1, 5.2 and 5.3, there exists a genus-2 SBLF $f_s : M_s \rightarrow S^2$ which has a section with self-intersection 0 and satisfies $W_{f_s} = W_s$.

Lemma 5.4. $\pi_1(M_s) \cong \mathbb{Z}$. Moreover, a generator of the group $\pi_1(M_s)$ is represented by the simple closed curve c_1 in the regular fiber.

Proof. Since f_s has a section, it can be shown by Van Kampen's theorem that the group $\pi_1(M_s)$ is isomorphic to the fundamental group of the union of the higher side and the round cobordism of f_s . In particular, we obtain:

$$\pi_1(M_s) \cong \begin{cases} \pi_1(\Sigma_2) / \langle c_5, \beta_1, \beta_{-1}, \gamma_1, \gamma_2 \rangle & (s = 2), \\ \pi_1(\Sigma_2) / \langle c_5, \alpha_1, \alpha_2, \dots, \alpha_{s-2}, \beta_{s-1}, \beta_{-1}, \gamma_1, \gamma_2 \rangle & (s \geq 3). \end{cases}$$

We identify the group $\pi_1(M_s)$ with the above quotient group via the isomorphism. The group $\pi_1(\Sigma_2)$ is generated by the elements $\lambda_i \in \pi_1(\Sigma_2)$ ($i = 1, 2, 3, 4$), where λ_i is represented by the loop Λ_i which is described as shown in Figure 11.

Since c_5, β_{-1} and γ_1 are free homotopic the loops $\lambda_3, \lambda_3 \cdot \lambda_4^{-1}$ and $\lambda_3^{-1} \cdot \lambda_4^{-1} \cdot \lambda_2$, respectively. So the elements λ_2, λ_3 and λ_4 vanish in the group $\pi_1(M_s)$. The loops α_i, β_j and γ_k are free homotopic to loops represented by words which consist of only $\lambda_2, \lambda_3, \lambda_4$ and their inverse. Thus, $\pi_1(M_s)$ is generated by the element λ_1 . In particular, we obtain $\pi_1(M_s) \cong \mathbb{Z}$. The remaining statement holds since c_1 is free homotopic to the loop λ_1 □

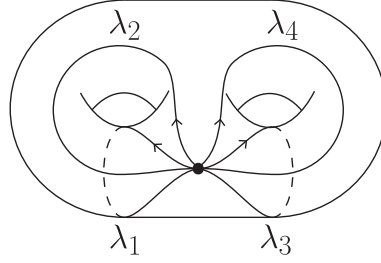


FIGURE 11.

By Lemma 5.4, we obtain the simply connected 4-manifold \tilde{M}_s from M_s by the following construction:

- Step.1 We first remove from M_s the interior of the regular neighborhood of the regular fiber F_0 of f_s in the lower side.
- Step.2 The boundary of $M_s \setminus \text{Int}(\nu F_0)$ is the trivial torus bundle over the circle. In particular, we obtain,

$$\pi_1(\partial(M_s \setminus \text{Int}(\nu F_0))) \cong \pi_1(S^1) \times \pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}^2.$$

Moreover, a certain primitive element μ in the group $\pi_1(T^2) \cong \mathbb{Z}^2 \in \pi_1(\partial(M_s \setminus \text{Int}(\nu F_0)))$ is mapped to the generator of $\pi_1(M_s \setminus \text{Int}(\nu F_0)) \cong \mathbb{Z}$ by the natural homomorphism $\pi_1(\partial(M_s \setminus \text{Int}(\nu F_0))) \rightarrow \pi_1(M_s \setminus \text{Int}(\nu F_0))$. We can attach νF_0 to $M_s \setminus \text{Int}(\nu F_0)$ so that the attaching circle of the 2-handle of νF_0 is along the simple closed curve which represents $([S^1], \mu) \in \pi_1(\partial(M_s \setminus \text{Int}(\nu F_0)))$, where $[S^1] \in \pi_1(S^1)$ is the generator. We denote by \tilde{M}_s the 4-manifold obtained by the above attachment.

In other word, we obtain the manifold \tilde{M}_s from M_s by logarithmic transformation on F_0 with multiplicity 1. In particular, the BLF structure on $M_s \setminus \text{Int}(\nu F_0)$ is naturally extended to that on \tilde{M}_s . We denote this SBLF by $\tilde{f}_s : \tilde{M}_s \rightarrow S^2$. We also remark that Kirby diagrams of M_s and \tilde{M}_s can be drawn as shown in Figure 12 and 13, respectively, by using the method in [3]. The 2-handles corresponding to the vanishing cycles $\alpha_1, \dots, \alpha_{s-2}$ is in the shaded parts in Figure 12 and Figure 13. These parts are empty if s is equal to 2.

The following theorem states that \tilde{f}_s gives the explicit example of genus-2 SBLF structure on $\sharp(s-2)\mathbb{CP}^2$.

Theorem 5.5. *For each $s \geq 2$, \tilde{M}_s is diffeomorphic to the manifold $\sharp(s-2)\mathbb{CP}^2$.*

Proof. We prove this theorem by Kirby calculus. A Kirby diagram of \tilde{M}_s is shown in Figure 13, where the framing of the 2-handles drawn in the bold curves is 0, the framing of the 2-handle of νF_0 is described by the broken curve and the framings of the other 2-handles are all -1 . We obtain Figure 14 by sliding several 2-handles to the 2-handle of the round 2-handle and isotopy moves give Figure 15. We next slide the 2-handles corresponding to the vanishing cycles $\alpha_1, \dots, \alpha_{s-2}$ to the 2-handle of the round 2-handle. We can eliminate the obvious canceling pair and we obtain Figure 16. Sliding the 2-handle corresponding to γ_2 to that corresponding to γ_1 gives Figure 17. We get Figure 18 by isotopy moves. By sliding the 2-handle corresponding to γ_1 to that corresponding to β_{-1} , we obtain Figure 19. Isotopy moves give Figure 20 and Figure 21 is obtained by sliding the 2-handle corresponding to β_{-1} to the 0-framed meridian and isotopy moves. The diagram described in Figure 21 can be divided

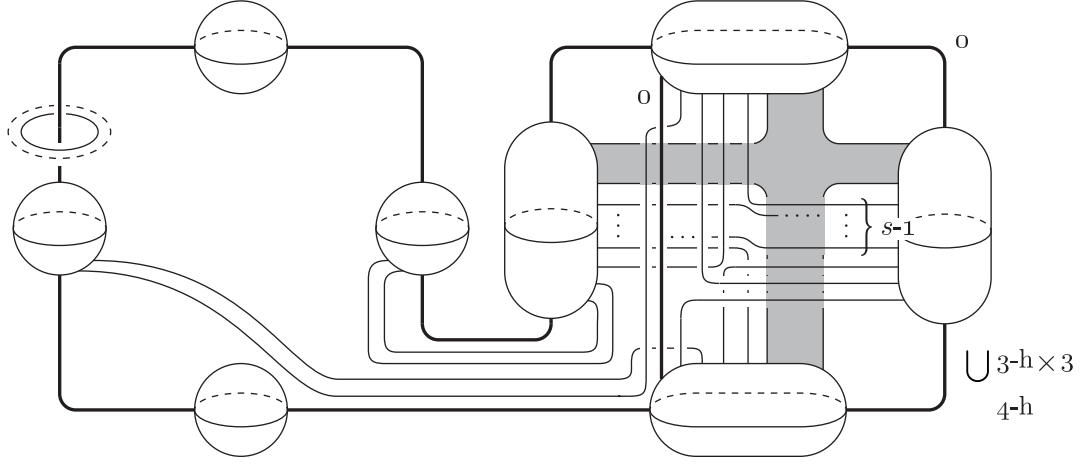
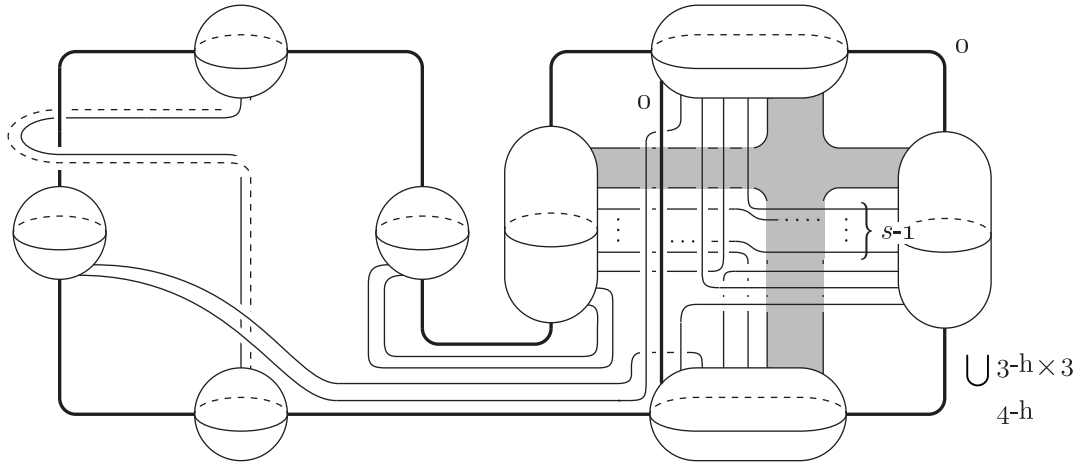
into two components. We look at the left component in Figure 21. By sliding the outer 2-handle to the 2-handle of νF_0 , we can change the left component in Figure 21 into the diagram as described in Figure 22. Isotopy moves give Figure 23 and Figure 24 and we obtain the diagram in the left side of Figure 25 by canceling the obvious canceling pair. To obtain the diagram in the right side of Figure 25, we slide the -1 -framed 2-handle to the 1 -framed 2-handle and then eliminate the canceling pair. Since the handle decomposition of \tilde{M}_s has three 3-handles, the 0 -framed unknots in Figure 25 can be eliminated. Eventually, we can change the diagram in Figure 21 into the diagram in Figure 26. The diagram in Figure 27 is same as in Figure 26, but the 2-handles in the shaded part are described in Figure 27. By isotopy moves, we get Figure 28. This diagram is similar to Figure 24 in [10]. By using similar technique to [10], we can change the diagram in Figure 28 into $s - 2$ unknots with -1 framing. This completes the proof of Theorem 5.5. □

As a result of Theorem 5.5, Theorem 1.5 holds.

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FIGURE 12. a Kirby diagram of M_s FIGURE 13. a Kirby diagram of \tilde{M}_s

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-0043,
JAPAN

E-mail address: k-hayano@cr.math.sci.osaka-u.ac.jp

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-0043,
JAPAN

E-mail address: m-sato@cr.math.sci.osaka-u.ac.jp

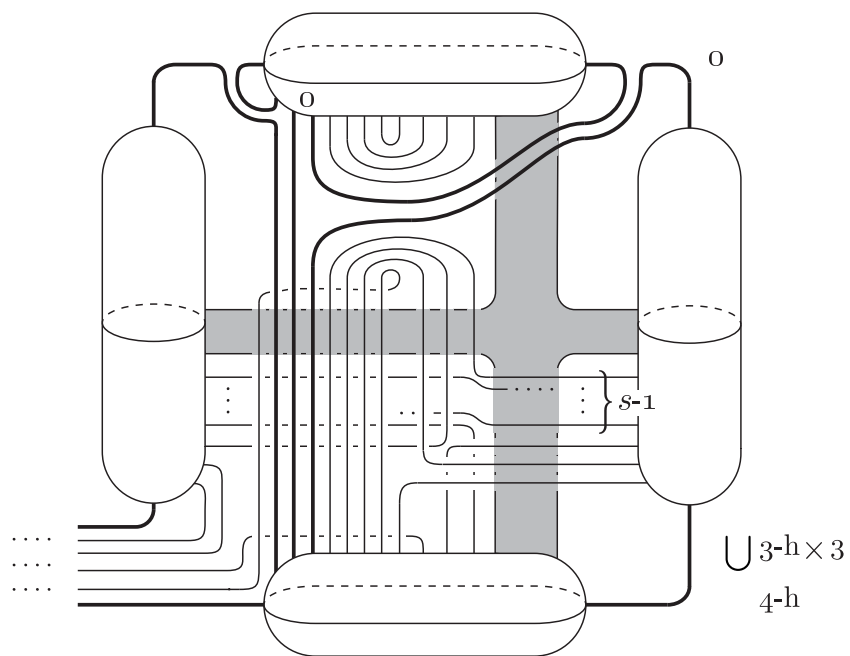


FIGURE 14.

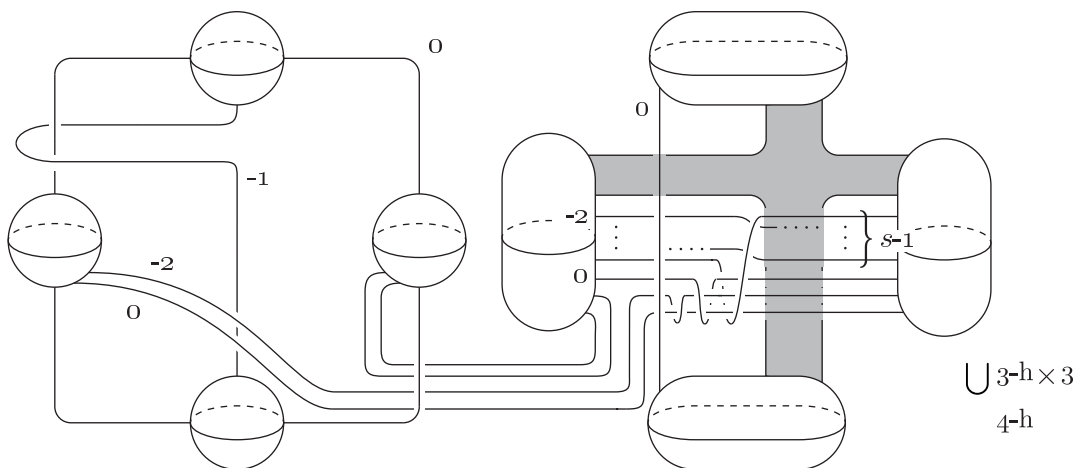


FIGURE 15.

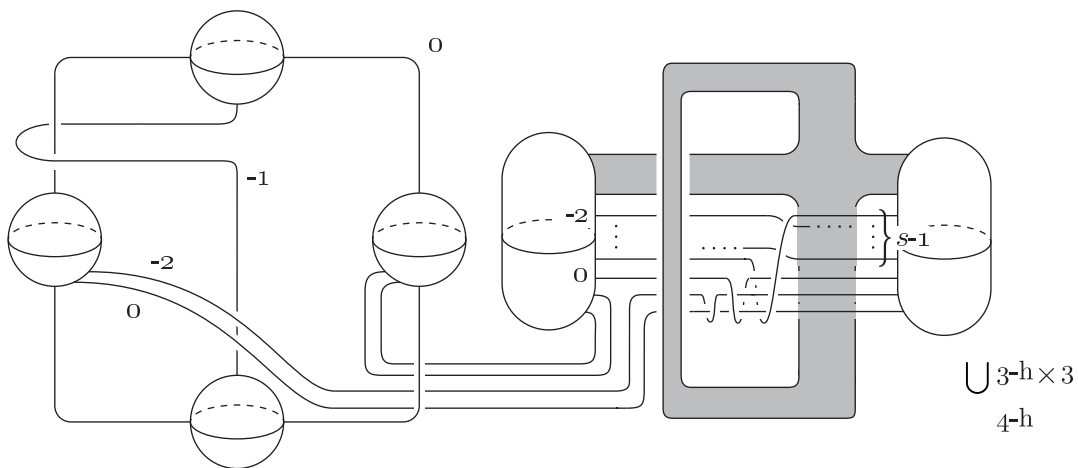


FIGURE 16.

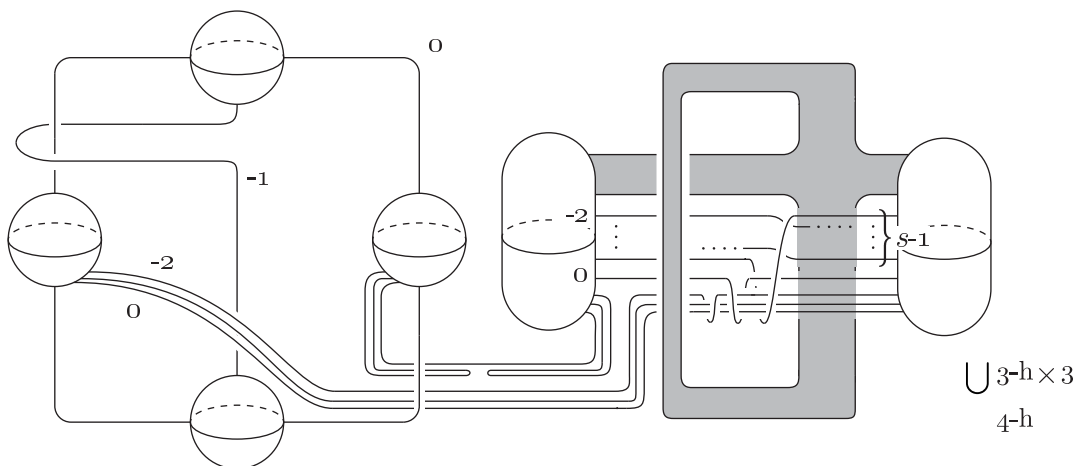


FIGURE 17.

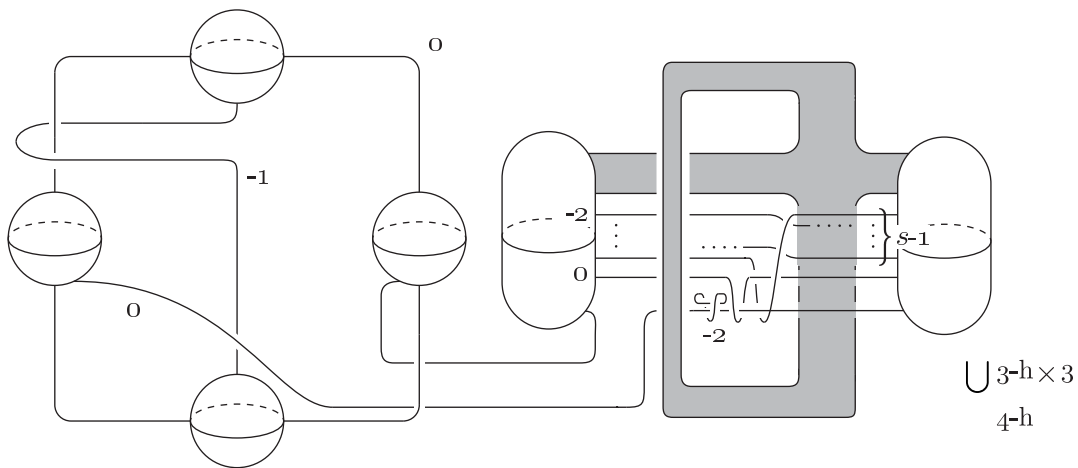


FIGURE 18.

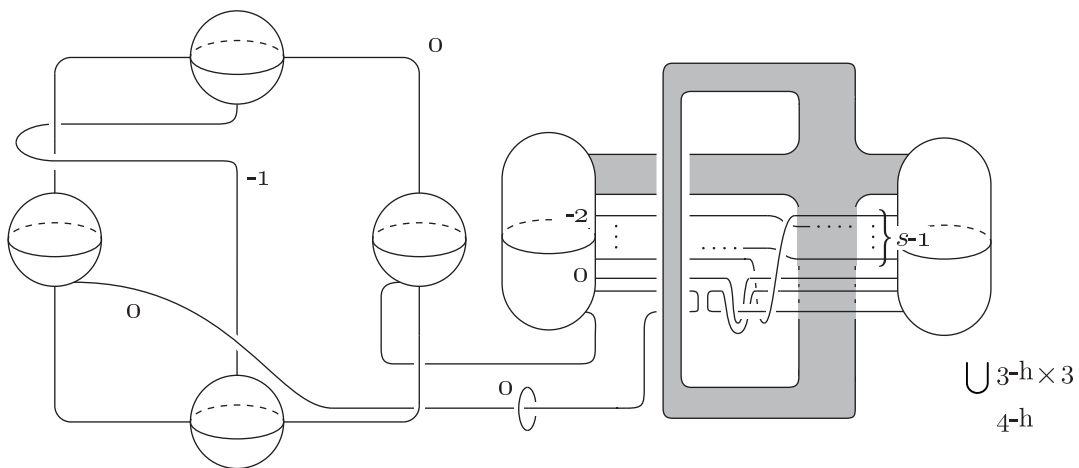


FIGURE 19.

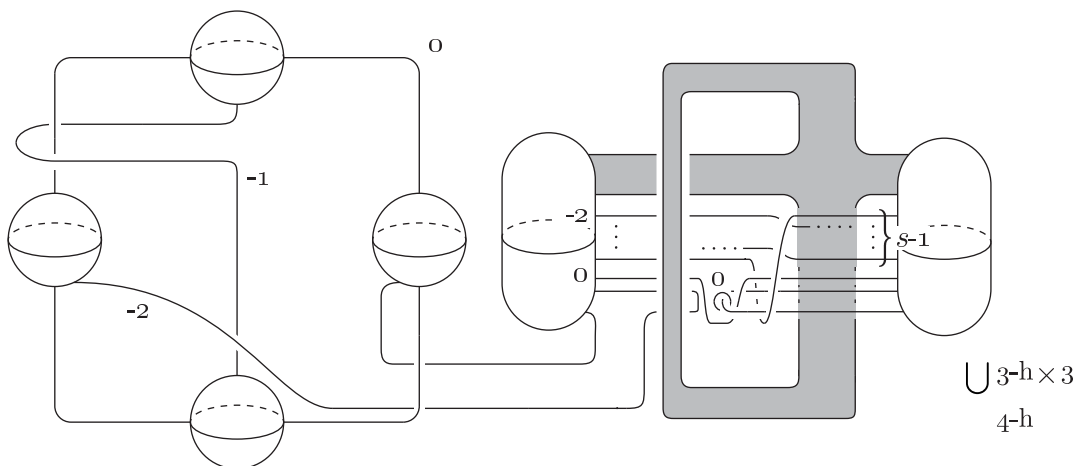


FIGURE 20.

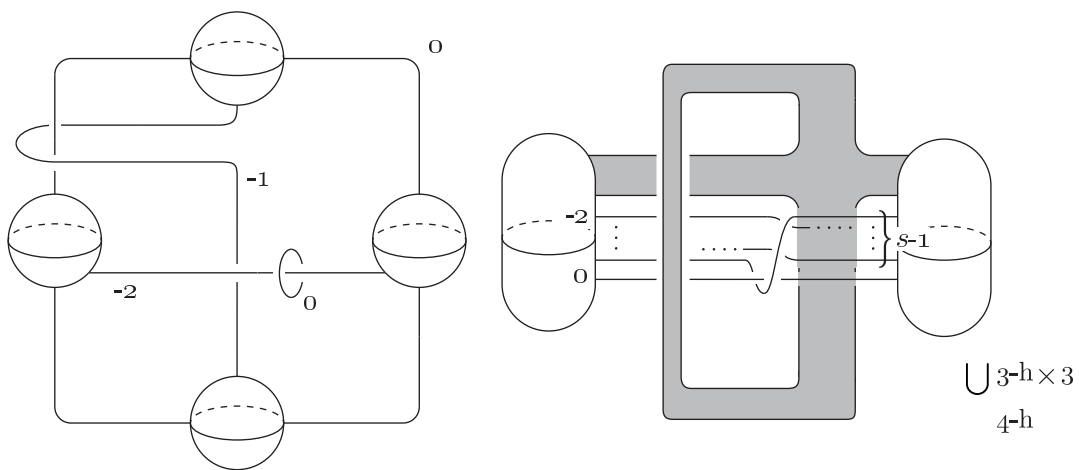


FIGURE 21.

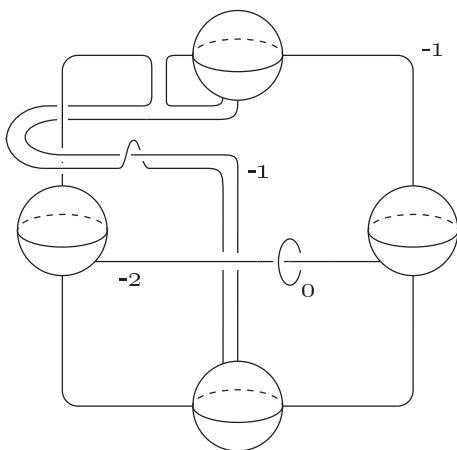


FIGURE 22.

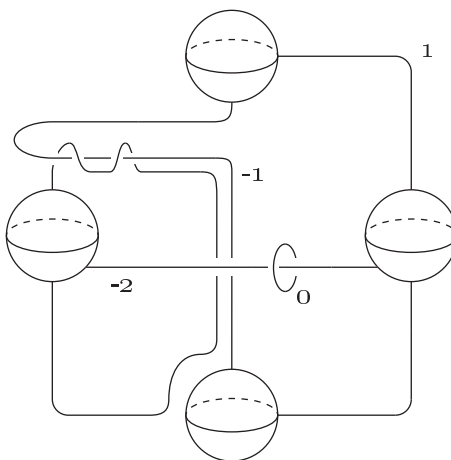


FIGURE 23.

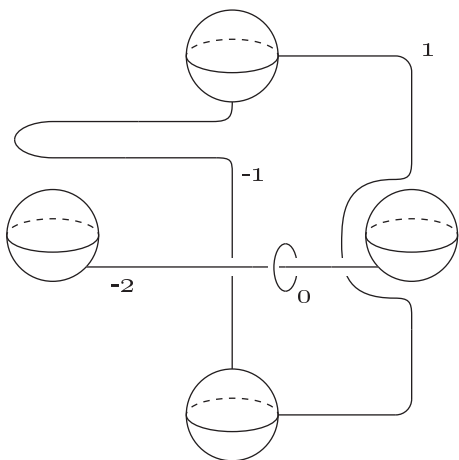


FIGURE 24.

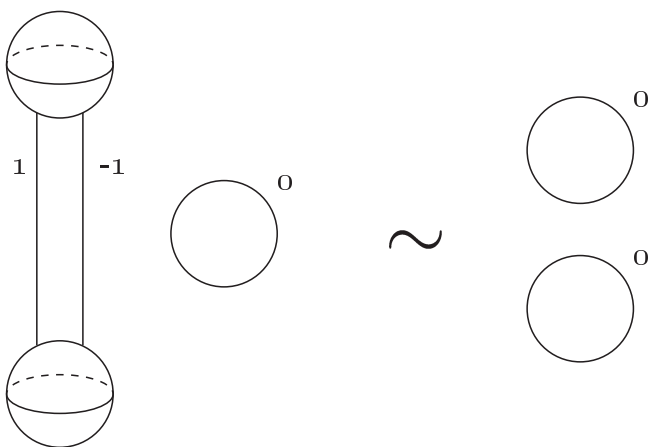


FIGURE 25.

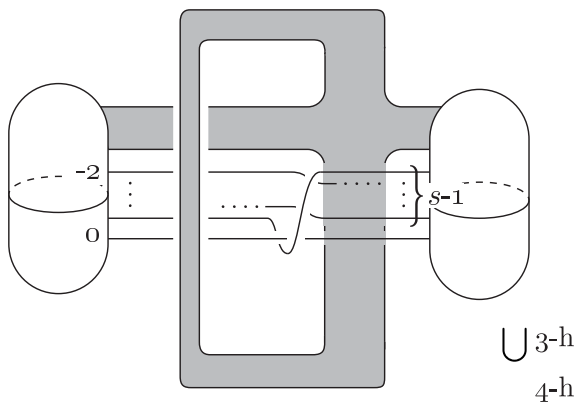


FIGURE 26.

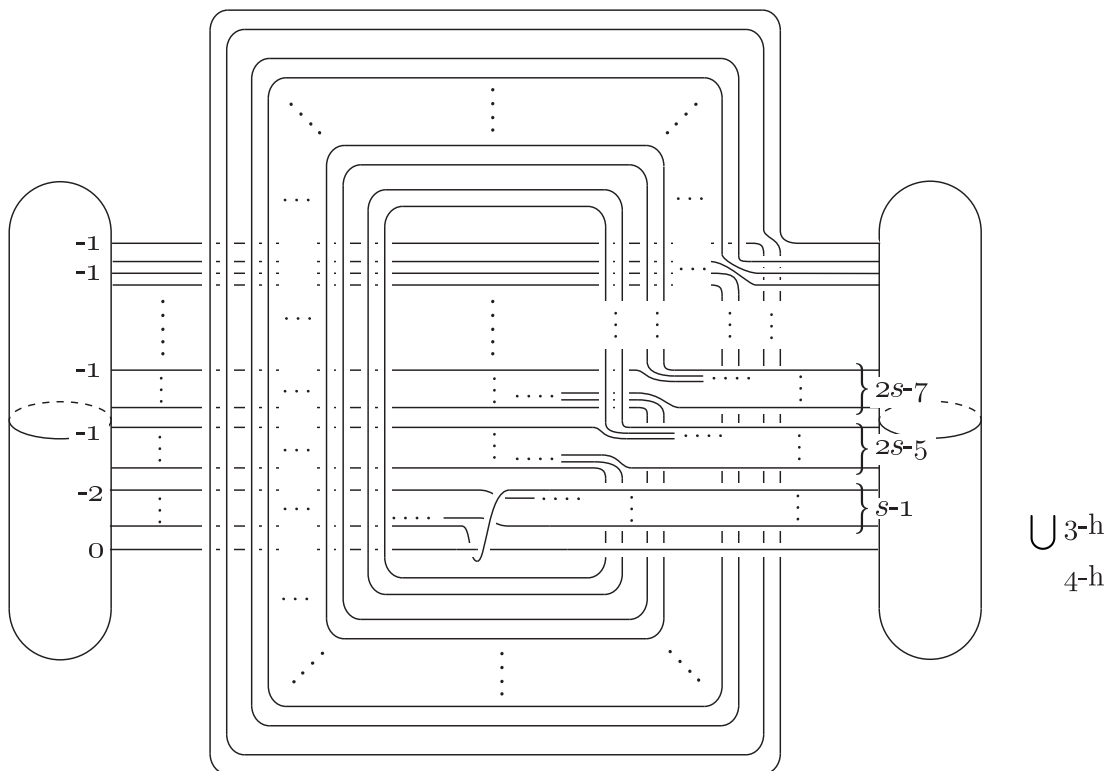


FIGURE 27.

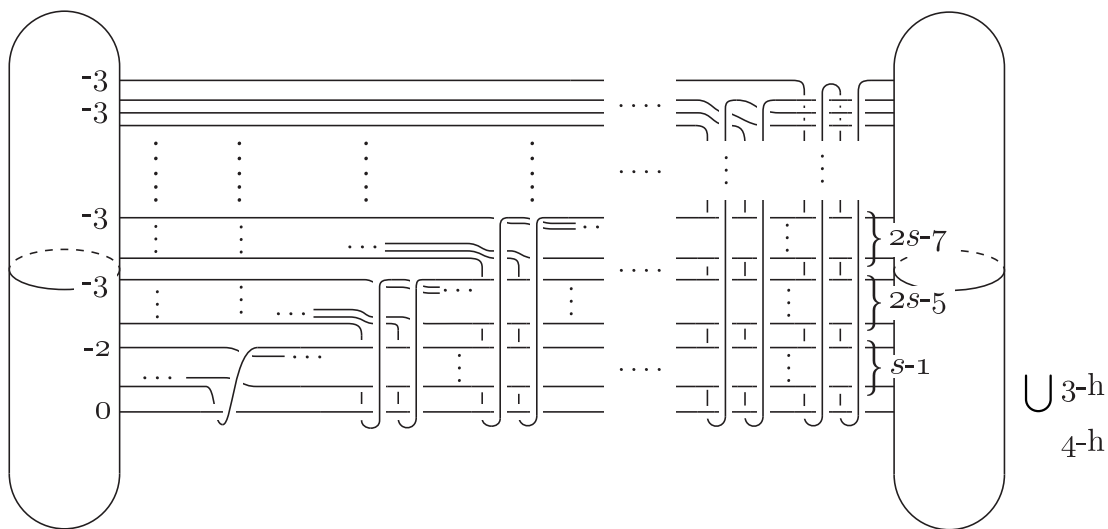


FIGURE 28.